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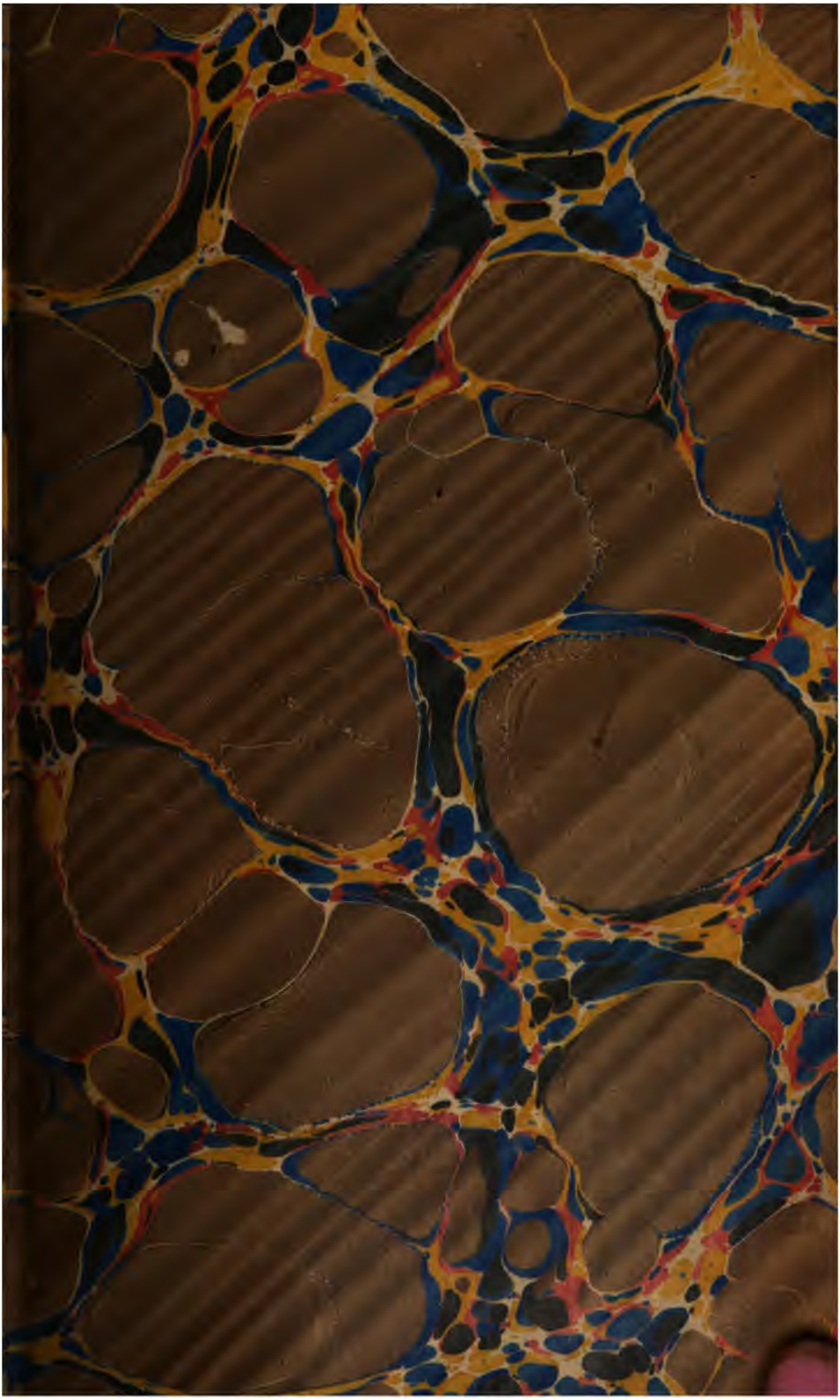

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NEW SERIES

OF THE

MATHEMATICAL

REPOSITORY.

By THOMAS LEYBOURN,
Of the Royal Military College.

VOL II.

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A Memoir on Elliptic Transcendentals. By ADRIEN MARIE LE GENDRE, Member of the National Institute of France, and

03-27-40 MLC

Directions for the Binder.

I. The Binder is desired to observe, that the vol. consists of four different sets of signatures. The first set, viz. A, B, C, and D, contains the questions, and is to be placed at the beginning of the vol. to serve as a table of contents to the first part.

II. The first, second, and third parts of the vol. must follow in their order. The two sets of questions of vol. 3. (viz. the signatures A and B) which are stiched up with the numbers forming the second vol. must be preserved apart till the third vol. be completed.

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NEW SERIES

OF THE

MATHEMATICAL REPOSITORY.

I. MATHEMATICAL QUESTIONS,

To be Anſwered in Number VI.

I. QUESTION 121, *by* SCOTICUS.

If α and β denote given arches, and
$\text{ſec.}\,\alpha\ \text{ſec.}\,\beta + \text{tan.}\,\alpha\ \text{tan.}\,\varphi = \text{ſec.}\,\beta.$
Required the values of ſec. φ and tan. φ?

II. QUESTION 122, *by* AMICUS.

If two tangents PE, PF, be drawn to a conic ſection, and the line EF joining the points of contact be produced to meet the directrix in G; then if PI be drawn through the focus to meet EF in I, EI will be to IF as EG to FG, required the demonſtration?

III. QUESTION 123, *by* Mr. JAMES WHALLEY, *Bolton.*

To conſtruct a trapezium there is given one of the ſides, both the diagonals and the angle formed by their interſection, and the angle made by either diagonal and ſide oppoſite to the given one?

IV. QUESTION 124, *by* SENEX.

Let there be two given points in the circumference of a circle given in magnitude and poſition: a circle may be found, given alſo in magnitude and poſition, ſuch, that if from any point in its circumference ſtraight lines be drawn to the given points, and a tangent to the given circle; the ſquare of the tangent ſhall have to the rectangle of the lines drawn to the given points a given ratio?

V. QUESTION 125, *by Mr.* JOSEPH BREWER, *Preſton.*

Two right lines being given by poſition, a circle may be found, ſuch, that if another circle be deſcribed upon any radius thereof as a diameter, the chord of the arc of this latter circle intercepted by the lines given by poſition ſhall be of a given length?

VI. QUESTION 126, *by Mr.* J. JOHNSON.

Given the poſition of the centres of gravity of the three triangles, formed by lines drawn from the angular points of a triangle to the centre of the inſcribed circle; to determine the poſition of the triangle?

VII. QUESTION 127, *by* HYPATIA.

Suppoſe $u = \frac{a^{a-n} - x^{x-n}}{a-x}$, what is the value of u when $= x\ a$?

VIII. QUESTION 128, *by* URSA MINOR.

Let AB, AC, be two ſtraight lines given by poſition, and let a ſtraight line DE meet them in D and E ſo that AD + AE is equal to a given line, and let DE be divided at V, ſo that DV may have to VE a given ratio. Required the locus of the point V?

IX. QUESTION 129, *by Mr.* CUNLIFFE, *R. M. College.*

To find two rational numbers, ſuch, that if a given number n be ſubtracted from either of them or from their ſum or difference, the four remainders ſhall all be rational ſquares?

X. QUESTION 130, *by* TRIGONOMETRICUS.

It is required to demonſtrate that the tangent of an arch of 9° is equal to $1+\sqrt{5}-\sqrt{(5+2\sqrt{5})}$, the radius being 1?

The next Six Queſtions are ſelected from the Cambridge Univerſity Calendars for 1804 *and* 1805.

XI. QUESTION 131, *from the* CAMB. UN. CAL.

A given parabolic conoid is put into a fluid, and when it is in a poſition of permanent equilibrium the baſe is wholly extant, and inclined to the horizon; having given its ſpecific gravity relatively to that of the fluid, it is required to determine the dimenſions of the part immerſed?

XII. QUESTION 132, *from the* CAMB. UN. CAL.

Find the centre of gravity of a cylindrical portion of the atmoſphere meaſured from the earth's ſurface, the force of gravity being conſtant?

XIII. QUESTION 133, *from the* CAMB. UN. CAL.

Required the nature of the curve along which a heavy body, deſcending by the force of gravity, will preſs upon the curve at any point, with a force proportional to the ordinate at that point?

XIV. QUESTION 134, *from the* CAMB. UN. CAL.

Let the curve AP*p* be generated by taking the ordinate MP always equal to the difference of the chord AC, and the verſed ſine AM, of the circular arc AC. It is required to determine the greateſt ordinate of the curve, and alſo its area?

XV. QUESTION 135, *from the* CAMB. UN. CAL.

Determine the apparent magnitude of a ſtraight rectilinear object, placed at a given depth, parallel to a ſurface of water; the eye being ſituated at any point in the plane paſſing through the object perpendicular to the ſurface?

XVI. QUESTION 136, *from the* CAMB. UN. CAL.

Let the roots of the equation $x^n - px^{n-1} + qx^{n-2} -$ &c. $= 0$, be $a, b, c,$ &c. and thoſe of the equation $nx^{n-1} - (n-1)px^{n-2} + (n-2)qx^{n-3} -$ &c. $= 0$, be $\alpha, \beta, \gamma,$ &c.; then, if when $\alpha, \beta, \gamma,$ &c. are ſucceſſively ſubſtituted in the equation $x^n - px^{n-1} + qx^{n-2} -$ &c. $= 0$, the reſults are P, Q, R, &c. and when $a, b, c,$ &c. are ſubſtituted in the equation $nx^{n-1} - (n-1)px^{n-2} + (n-2)qx^{n-3} -$ &c. $= 0$, the reſults are $p, q, r,$ &c. it will be as $P \times Q \times R$, &c. : $p \times q \times r$, &c. :: $1 : n^n$. Required the demonſtration?

XVII. QUESTION 137, *by Mr.* HENRY LIGHTBOWN.

Suppoſe a heavy flexible chain of a given length having its ends faſtened together to be thrown over the vertex of a given ſquare pyramid whoſe ſurface is perfectly poliſhed; what portion of the ſurface will be included between the vertex and the chain when it has deſcended as low as it can get?

XVIII.

XVIII. QUESTION 138, *by Mr.* THOS. BAZLEY, *Bolton.*

Suppoſe a perfectly flexible chain of a given length to have one end faſtened to an immoveable tack, and a ring at the other end to be put upon a perfectly ſmooth inflexible line or rod given by poſition; it is required to determine that point upon the rod at which the ring will reſt?

XIX. QUESTION 139, *by Mr.* LOWRY.

In a given trapezium to inſcribe another, ſuch that three of its ſides may paſs through given points, and the remaining ſide be of a given length?

XX. QUESTION 140, *by Mr.* W. SIMPSON, *Bolton.*

$$n \times \left\{ (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4 + \&c.\ \textit{ad infinitum} \right\}$$

$$= \frac{x^n - 1}{x^n} + \tfrac{1}{2}\left(\frac{x^n - 1}{x^n}\right)^2 + \tfrac{1}{3}\left(\frac{x^n - 1}{x^n}\right)^3 + \tfrac{1}{4}\left(\frac{x^n - 1}{x^n}\right)^4 + \tfrac{1}{5}\left(\frac{x^n - 1}{x^n}\right)^5$$

+ &c. *ad infinitum.* Required the demonſtration?

XXI. QUESTION 141, *by Mr.* CUNLIFFE.

It is required to find three whole numbers ſuch, that the ſum of every two may be a cube; and alſo the ſum of all the three a cube?

XXII. QUESTION 142, *by Mr.* CUNLIFFE.

P is a given point in the diameter AB produced of a given circle, let any line PCD be drawn cutting the circle in C and D, and let AC, BD, be drawn interſecting in I: now, if in BD produced there is continually taken DR = DI, required the equation and quadrature of the curve which is the locus of R?

XXIII. QUESTION 143, *by Mr.* BAZLEY.

Suppoſe a thread of a given length to be faſtened at one end to a given point on a perfectly ſmooth horizontal plane having a ſtraight edge; and a weight attached to the other end of the thread and brought to the edge of the horizontal plane with the thread ſtretched; then if the weight be ſuffered to deſcend by the force of gravity from this poſition; it is required to determine the time in which the thread will acquire any other poſſible aſſigned poſition upon the plane, together with the velocity of the deſcending weight in ſuch poſition; alſo required the nature of the curve deſcribed by the deſcending weight?

XXIV. QUESTION 144, *by Mr.* CUNLIFFE.

Having given a triangle, it is required to determine a circle and concentrical ellipſe, together with the poſition of a right line paſſing through the ſaid centre, ſuch, that one of the angles of the triangle being always placed upon the right line, the other two angles ſhall be one in the circumference of the circle, and the other in the periphery of the ellipſe?

XXV. QUESTION 145, *by Mr.* I. I.

If from a point without an ellipſe or an hyperbola, two tangents be drawn to the curve, and likewiſe two ſtraight lines be drawn to the foci, the angle contained by one tangent and the line drawn to one focus will be equal to the angle contained by the other tangent and the line drawn to the other focus?

XXVI. QUESTION 146, *by Mr.* LOWRY.

Let there be any number of ſtraight lines AP, BP, CP, DP, &c. given by poſition, and let Z be a given point without them; it is required to deſcribe a circle through the points P and Z, to interſect the given lines in A, B, C, D, &c. ſo that the ſum of all the intercepted parts AP, BP, CP, DP, &c. may be equal to a given line?

XXVII. QUESTION 147, *by Mr.* WM. WALLACE, *R. M. College.*

If x denote an arch of the meridian, and y denote the meridional parts correſponding to the latitude x, then

$$y = x + \frac{x^3}{6} + \frac{x^5}{24} + \frac{61x^7}{5040} + \frac{277x^9}{725678} + \&c.$$

and by reverting the ſeries

$$x = y - \frac{y^3}{6} + \frac{y^5}{24} - \frac{61y^7}{5040} + \frac{277y^9}{725678} - \&c.$$

where it is remarkable that the numeral coefficients of the powers of x are the ſame as thoſe of the like powers of y, and the ſigns of the terms of the one ſeries are all +, but thoſe in the other ſeries + and — alternately. Shew the reaſon of theſe peculiarities?

XXVIII. QUESTION 148, *by Mr.* I. R.

Let two circles, one of which is included within the other, be given in a plane, and let there be given a point, ſituated within both the circles, in the line joining the two centres; it is required to deſcribe a circle, from the given point as a centre, to meet the peripheries of the given circles, ſo that the angle contained by two ſtraight lines drawn from the given point to the two points of concourſe may be the leaſt poſſible?

XXIX. QUESTION 149, *by* MECHANICUS.

To place a ſtraight rod, or beam, loaded with any given weights, ſo that it may be in equilibrio, reſting on an upright prop, and one end touching a ſmooth vertical wall given by poſition?

XXX. PRIZE QUESTION 150, *by* LIMENUS.

Let A and B be two given points without a circle given in magnitude and poſition; a point C may be found within the circle ſuch that if any ſtraight line be drawn through it cutting the circle in F and G, and AF, BG, be joined, the ſquare of AF will have to the ſquare of BG the ratio which is compounded of the ratio of FC to CG, and a certain given ratio, and which ratio is alſo to be found?

Solutions to theſe Queſtions muſt come to hand (poſt-paid) by the firſt of February, 1806.

NEW SERIES
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II. MATHEMATICAL QUESTIONS
To be Answered in Number VII.

I. QUESTION 151, *by* PAPPUS.

Given the length of a circular arc, to find the length of the chord connecting its extremities, ſo that the area of the included ſegment may be the greateſt poſſible?

II. QUESTION 152, *by* HERMODORUS.

To inſcribe ſeven equal circles in a given circle?

III. QUESTION 153, *by* CENTROBARICOS.

The radius of the greater end of the fruſtum of a paraboloid being R, that of its leſs end r, and the height of the fruſtum h; it is required to give a neat expreſſion for the poſition of its centre of gravity?

IV. QUESTION 154, *by* DIOPHANTUS.

Find a trapezium which can be inſcribed in a circle ſuch, that its ſides and area may be expreſſed by whole numbers?

V. QUESTION 155, *by Mr.* J. JOHNSON.

In a given ſpherical triangle, it is required to inſcribe another, ſo that its perimeter may be the leaſt poſſible?

VI. QUESTION 156, *by* LIMENUS.

If a ſtraight line be drawn to meet the four ſides of a trapezium in a, b, c, d, and the two diagonals in m and n; then will ab be to cd as $am \times nb$ to $cm \times nd$. Required the demonſtration?

VII. QUESTION 157, *by* J. W. LEEDS.

Let A be a given point in the periphery of a given circle, let any chord AC be drawn, and divided in the point E, so that the rectangle AC × CE ſhall be equal to a given ſpace; it is required to determine the equation and quadrature of the locus of the point E?

VIII. QUESTION 158, *by* JUVENIS.

Given $2x\dot{x} - 3y\dot{x} = 10\dot{y} - 5x\dot{y}$, to find the equation of the fluents?

IX. QUESTION 159, *by* A. M.

If from the extremities A and B of the diameter AB of a given ſemicircle ABC, two chords, AC, BD, be drawn interſecting each other in P, and BD biſecting the arc AC in D; it is required to determine the equation and quadrature of the curve which is the locus of the point P?

X. QUESTION 160, *by Mr.* CUNLIFFE, *R. M. College.*

Required the ſum of the infinite ſeries

$$1 - \tfrac{1}{2} \times (\tfrac{1}{3}) + \tfrac{1}{3} \times (\tfrac{1}{3} + \tfrac{1}{5}) - \tfrac{1}{4} \times (\tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7}) + \tfrac{1}{5} \times (\tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{9}) \text{ \&c.}$$

XI. QUESTION 161, *by Mr.* W. SIMPSON, *Bolton*,

Let ACB be a given triangle, D, L, Q given points in the ſide AB; through D draw any right line DEF, cutting AC, BC, in E and F reſpectively, and through E and F draw right lines LEP, QFP interſecting in P. Required the locus of P?

XII. QUESTION 162, *by* AMICUS.

If a given weight be ſuſtained by means of a ſtring whoſe ends are connected (by loops or rings) to two rods placed in the ſame vertical plane, and making given angles with the horizon; it is required to aſcertain the poſition of the ſaid weight when it reſts in equilibrio, ſuppoſing it has liberty to ſlide freely along the ſtring?

XIII. QUESTION 163, *by Mr.* CUNLIFFE.

Find ſuch poſitive values of x, y, and z, in whole numbers, as will make the formulæ
$x^2 + xy + y^2$, $x^2 + xz + z^2$ and $y^2 + yz + z^2$,
all rational ſquares?

XIV. QUESTION 164, *by* URSA MINOR.

A ſtraight line drawn through the focus of any conic ſection, to the point in which a diameter meets the directrix, will be perpendicular to a tangent at either extremity of that diameter?

XV. QUESTION 165, *by* AMICUS.

If one end of a given beam reſt againſt a ſmooth vertical plane, and the other end be ſuſtained by a weight faſtened to a ſtring, which paſſes over a pulley placed at a given point; it is required to determine the poſition of the beam when it reſts in equilibrio?

XVI. QUESTION 166, *by* X. Y.

Suppoſe two weights connected by a ſtring, which paſſes over a pulley, and that one of the weights is placed upon an inclined plane, while the other hangs in the air by the connecting ſtring. It is required to determine the *locus* of the centre of gravity of the weights, ſuppoſing the relative poſition of the pulley and plane, the length of the ſtring, and the weights to be all given?

XVII. QUESTION 167, *by* LIMENUS.

When any number of circles paſs through the ſame two points, tangents drawn from any point in either of the two external circumferences to the remaining circles ſhall be to one another in a given ratio?

XVIII QUESTION 168, *by* LAPUTIENSIS.

Suppoſe two bodies whoſe weights are known to be connected by a flexible line or ſtring paſſing through a ſmall hole in a perfectly ſmooth horizontal plane, and let the body which is above the plane be projected thereon, with a given velocity at a given diſtance from the hole, in a direction perpendicular to the ſtring which is kept tight by the other body hanging freely; it is required to determine the velocity, time, and angle formed by the ſtring with the firſt poſition, when the projected body has attained any other poſſible aſſigned diſtance from the hole through which the ſtring paſſes?

XIX. QUESTION 169, *by* HERMODORUS.

Given the vaniſhing line, its centre and diſtance, to find the perſpective repreſentation of a circle from the given repreſentation of one radius, or of an inſcribed triangle?

XX. QUESTION 170, *by* HERMODORUS.

Given in poſition a circle and two points; if from the two points be inflected two ſtraight lines to any point of the circle, and from the other two points in which theſe lines meet the circle, there be inflected two other ſtraight lines to a point in the circle, ſo that one of them may be parallel to a ſtraight line given by poſition, then the other ſtraight line will either tend to a given point, or make a given angle with a ſtraight line tending to a given point?

XXI. QUESTION 171, *by* ANALYTICUS.

Find a ſeries for the *n*th power of the logarithm of a number?

XXII. QUESTION 172, *by* SENEX.

It is required to find the nature or equation of a curve ſuch, that any right line RPQ being drawn through a given point P, to cut the curve in two points R and Q, and tangents from theſe points produced to interſect each other in I, the angle QIR may always be of the ſame conſtant magnitude, that is always equal to a given angle?

XXIII. QUESTION 173, *by Mr.* JOHN FLETCHER.

If the magnitude of a circle is given; it is required to find the poſition of a right line RS, and that of a point P in another line given in poſition, from whence if PAB be drawn to cut the circle in A and B, and perpendiculars AR and BS are drawn to RS, their rectangle will be always a given magnitude?

XXIV. QUESTION 174, *by* AMICUS.

If a given cone, ſuſpended by its vertex, be made to revolve with its axis in a vertical plane; it is required to determine the force exerted on the centre of ſuſpenſion at any given poſition of the revolving body?

XXV. QUESTION 175, *by Mr.* WALLACE, *R. M. College.*

Suppoſe a quadrilateral figure to be inſcribed in another quadrilateral figure, agreeable to the conditions required in queſtion 120 of the New Series of the Repoſitory, then the oppoſite ſides of the inſcribed figure when produced, will interſect each other in the diagonals produced of the circumſcribed figure, Required the demonſtration?

XXVI. QUESTION 176, *by* MARLOVIENSIS.

It is well known that the ſum of the ſeries $x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \frac{x^5}{5^2} -$ &c. *ad infinitum*, is expreſſed by the fluent of $(x^{-1}\dot{x} \times \text{fl.} \frac{\dot{x}}{1+x})$, or area of a tranſcendent hyperbola. Under what circumſtances, can the ſame ſum be exhibited by the multiples or powers of the arc of a circle, and how is this to be inveſtigated ?

XXVII. QUESTION 177, *by* MARLOVIENSIS.

It is required to ſhew a ſimple practical method (capable of demonſtration), of putting a globe or ſphere into perſpective; having given the magnitude and ſituation of the ſphere, the height of the eye above the ground plane, and its diſtance from the plane of projection, which is. as uſual, ſuppoſed to be perpendicular to the horizon, alſo to ſhew what ſituation the globe muſt have, that the contour of its perſpective may be any propoſed conic ſection ?

XXVIII. QUESTION 178, *by* MECHANICUS.

If two given bodies A and B, connected by a ſtring, be ſupported on an inclined plane, by means of the ſtring's paſſing through a ring at A, and any given force be impreſſed on the body A, in a horizontal direction ; it is required to determine the nature of the curve deſcribed by the body, and the circumſtances of the motion ?

XXIX. QUESTION 179, *by Mr.* WALLACE, *R. M. College.*

Suppoſe two particles of matter p and q, to be connected by a ſtring pAq, and to reſt in equilibrio upon two inclined planes AB, AC, the ſtring being ſuppoſed to paſs over the common ſection of the planes at the point A; then, if a circle be deſcribed through the points p, A, q, and the particles without changing their poſition are diſengaged from the ſtring pAq, and connected by another paq, which paſſes over a peg faſtened at any point a in the circumference of the circle, the particles p and q will ſtill reſt in equilibrio upon the planes. Required the demonſtration ?

XXX. PRIZE QUESTION 180, *by* GEOMETRICUS.

If the oppoſite ſides of a hexagon, inſcribed in a circle, be produced till they meet, the three points of interſection will be in the ſame ſtraight line. Required the demonſtration ?

Solutions to theſe queſtions muſt come to hand (poſt-paid) by the Firſt of November, 1806.

NEW SERIES
OF THE
MATHEMATICAL REPOSITORY.

III. MATHEMATICAL QUESTIONS.

To be anſwered in Number VIII.

I. QUESTION 181, *by* YANTO.

Bacchus caught Silenus aſleep by the ſide of a full caſk, he ſeized the opportunity of drinking, which he continued for two-thirds of the time that Silenus would have taken to empty the whole caſk. After that, Silenus awakes, and drinks what Bacchus had left. Had they drank both together, it would have been emptied two hours ſooner, and Bacchus would have drank only half of what he left for Silenus. Required the time in which they would empty the caſk ſeparately?

II. QUESTION 182, *by* AMICUS.

Two given beams, or rods, loaded with weights, being placed on a horizontal plane, ſo that their lower extremities may be oppoſed to each other, and the other extremities be ſupported by two parallel vertical planes; it is required to find their poſition when they are in equilibrio, and the preſſure exerted againſt each plane?

III. QUESTION 183, *by* QUIDAM.

Given all the four ſides, and the difference of the parts into which one of the angles is divided by a diagonal; to conſtruct the trapezium and point out the limits.

IV. QUESTION 184, *by Mr.* JOHNSON, *Birmingham.*

A given quantity of gold is to be made into a cup in the form of a ſegment of a ſphere, and *n* tenths of an inch in thickneſs: what are the dimenſions of the cup when it is made ſo as to contain the greateſt quantity poſſible?

V. QUESTION 185, *by* QUIDAM.

Let a ſtring of a given length be faſtened at one end to a point A in the circumference of a circle, and wound tightly about the circumference, and brought with the other end T to meet the diameter AB in T: what muſt be the diameter of the circle that the area BTP included by the right lines TP, TB, and the arc BP may be a maximum, TP being the tangent from T?

VI. QUESTION 186, *by Mr.* JOHN WHITLEY.

If an ellipſe be circumſcribed by a triangle, and lines be drawn from the angles of the triangle to the points of contact, meeting the ellipſe again in three points; tangents drawn to the curve at theſe points will interſect each other in the ſaid lines produced. Alſo, if the ſaid tangents and the ſides of the triangle be produced till they meet, the three points of interſection will be in the ſame ſtraight line. Required the demonſtration?

VII. QUESTION 187, *by* LAPUTIENSIS.

Let two ſpherical balls given in magnitude and weight, and having their ſurfaces perfectly poliſhed, be put into a ſpherical veſſel of a given diameter, whoſe internal ſurface is alſo perfectly poliſhed: required the diſtances of the points of contact from the loweſt point of the veſſel, when the balls reſt in equilibrio?

VIII. QUESTION 188, *by Mr.* IVORY.

Two points being given in the ſame vertical plane, but not in one horizontal line; it is required to determine the poſition of two inclined planes ſuch, that the time of deſcent from the higher point to the lower, on theſe two planes, ſhall be leſs than on any other two inclined planes whatſoever?

IX. QUESTION 189, *by Mr.* CUNLIFFE.

Required the ſum of the infinite ſeries

$$\tfrac{1}{2}-\tfrac{1}{3}\times(\tfrac{1}{2})+\tfrac{1}{4}\times(\tfrac{1}{2}+\tfrac{1}{3})-\tfrac{1}{5}\times(\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4})+\tfrac{1}{6}\times(\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5})-\&c?$$

The next four Queſtions are ſelected from the Cambridge Univerſity Calendar for 1806.

X. QUESTION 190, *from the* CAMB. UN. CAL.

A given cylindric veſſel is ſupplied with water by a cock at a given rate. Then ſuppoſe, when the veſſel is full, that an aperture of a given area is made in its bottom; what is the loweſt point to which the ſurface of the water in the veſſel will deſcend, and alſo the time of its deſcent; the influx of the water by the cock being ſuppoſed ſlower than the efflux when the veſſel is full?

XI. QUESTION 191, *from the* CAMB. UN. CAL.

Bradley obſerved that every ſtar paſſed the meridian fartheſt ſouth when it came about ſix in the morning, whatever were its poſition with reſpect to the cardinal points of the ecliptic. Suppoſe then the place of a ſtar to be given; on what day will it paſs the meridian of London fartheſt to the ſouth; and at what hour will it paſs it on that day?

XII. QUESTION 192, *from the* CAMB. UN. CAL.

A perfectly flexible chain is wound round a cylinder ſupported with its axis parallel to the horizon. Then, if the weight and dimenſions of the cylinder be given, and alſo the length and weight of the chain; it is required to determine the time in which the chain, impelled by the force of gravity, will unwind itſelf; a given length being unwound at the commencement of the motion?

XIII. QUESTION 193, *from the* CAMB. UN. CAL.

A cylindrical veſſel, whoſe height is equal to its diameter, is filled with water; with what velocity muſt it be whirled round its axis, that half the water may be thrown out?

XIV. QUESTION 194, *by* AMICUS.

Of all the ellipſes that can be inſcribed in the ſame given quadrilateral, required that which approaches the neareſt to a circle?

XV.

XV. QUESTION 195, *by Mr.* I. R.

A parabola, and a point within it, being given in poſition, it is required to deſcribe another parabola, that ſhall paſs through the given point, touch the given parabola in a given point, and have its axis parallel to the axis of the given parabola.

XVI. QUESTION 196, *by Mr.* WILLIAM SLENDER.

Let there be a right line and a point given by poſition in the ſame vertical plane: and ſuppoſe a flexible ſtring of a given length, conſidered as without weight, to be faſtened at one end to the given point, and a ſmall heavy body or bead to ſlide freely thereon; required the locus of the bead, whilſt the other end of the ſtring is carried ſlowly along the right line given by poſition?

XVII. QUESTION 197, *by Mr.* IVORY.

Two points being given in a vertical plane, but not in the ſame horizontal line; it is required to determine the poſition of three inclined planes ſuch, that (ſuppoſing the body to paſs from one plane to another without any loſs of velocity) the time of deſcent may be leſs than the time on any other three planes, when the line of deſcent is of a given length?

XVIII. QUESTION 198, *by* MECHANICUS.

Required the nature of the curve along which a heavy body deſcending by the force of gravity, with a given initial celerity, will preſs upon the curve at any point with a force reciprocally proportional to the radius of curvature at that point?

XIX. QUESTION 199, *by Mr.* CUNLIFFE.

Find a ſeries of numbers, every one of which is diviſible into three ſuch parts, that if to the ſquare of each part the product of the other two be added, the three ſums thence ariſing ſhall all be rational ſquares?

XX. QUESTION 200, *by* HYPATIA.

Required the nature of the curve that is the locus of the point in which a perpendicular from the centre of an ellipſe meets a tangent to the ellipſe. Required alſo the rectification of the curve and its quadrature?

XXI. QUESTION 201, *by* CURIOSUS.

Find the ſhorteſt diſtance between two given points on the ſurface of a ſpheroid?

XXII. QUESTION 202, *by Mr.* WALLACE, *R. M. College*.

Find the length of a curve, the nature of which is expreſſed by the equation

$$e^{y} = \frac{e^{x}+1}{e^{x}-1},$$

where x and y denote the co-ordinates, and e the number of which the hyperbolic logarithm is unity?

XXIII. QUESTION 203, *by Mr.* CUNLIFFE.

To find three numbers ſuch, that the ſum of every two ſhall be a biquadratic; and the ſum of all the three a ſquare number?

XXIV. QUESTION 204; *by Mr.* IVORY, *R. M. College.*

If there be two concentric ellipſes, ſimilar to one another, and ſimilarly poſited, the proportion of the axes being that of $\sqrt{2}$ to 1; then if, from any point in the periphery of the inner ellipſe, any right line be drawn to terminate in the periphery of the outer ellipſe; the rectangle under the ſegments of that line, is equal to the ſquare of that ſemi-diameter of the inner ellipſe, which is parallel to the right line. Required the demonſtration?

XXV. QUESTION 205, *by Mr.* LOWRY.

Of all ſemi-parabolas having the ſame area, required that along which a body will deſcend, by the force of gravity, in the leaſt time poſſible?

XXVI. QUESTION 206, *by* URSA MINOR.

Suppoſe a cone to be cut by a plane, and the curved ſurface of the part cut off to be extended into a plane ſurface. Required the nature of the plane curve, formed by the common ſection of the cone and the cutting plane?

XXVII. QUESTION 207, *by Mr.* P. R.

Two magnetical poles being given in poſition; the force of each of which is ſuppoſed to be as the *mth.* power of the diſtance from it reciprocally; it is required to find a curve, in any point of which a needle (indefinitely ſhort) being placed, its direction, when at reſt, may be a tangent to the curve?

XXVIII. QUESTION 208, *by Mr.* WALLACE, *R. M. College.*

Shew that the rectification of the curve called the *line of ſines* depends upon that of an ellipſe, without employing the fluxionary calculus, or in any other way comparing the indefinitely little increments of the curves?

XXIX. QUESTION 209, *by Mr.* IVORY.

If ſtraight lines be inflected from the extremities of the tranſverſe axis of an ellipſe, or hyperbola, to any point in the curve, and perpendiculars, drawn to the inflected lines from the ſame point, be produced to cut the ſame axis, the part of the axis intercepted by the perpendiculars, will be conſtantly of the ſame magnitude. Required the demonſtration?

XXX. PRIZE QUESTION 210, *by Mr.* LOWRY.

In a given ellipſe to inſcribe a polygon of a given number of ſides, ſo that each ſide may paſs through a given point?

Solutions to theſe Queſtions muſt come to hand (poſt-paid) by the firſt of May 1807.

MATHEMATICAL REPOSITORY.

IV. MATHEMATICAL QUESTIONS.

To be Anſwered in Number IX.

I. QUESTION 211, *from the* BIJA GONETA, *or* HINDOO ALGEBRA.

Find two integer ſquare numbers whoſe ſum and difference diminiſhed by 1 ſhall both be ſquares?

"The *Bija Goneta* was written in Sanscrit by BHASKER ACHARIJ, a famous *Hindoo* Mathematician and Astronomer, who lived about the beginning of the 13th century of the *Christian* era: and was translated into Persian in 1634."

II. QUESTION 212, *by* INVESTIGATOR.

What ſum ſhould A, aged 30, pay on the event of his ſurviving, B, aged 65, for an annuity of 1000*l.* during their joint lives, uſing the Northampton table of the probable duration of life, and allowing 5 per cent. intereſt of money?

III. QUESTION 213, *by Mr.* BAZLEY.

Find a circle whereof the diameter, and the chord, ſine, and verſed-ſine of an arc of it, are all integers?

IV. QUESTION 214, *by* AMICUS.

The ends A, C of two given beams are moveable about two fixed points, and the other ends B, D are connected by a ſtring of a given length which paſſes over a fixed pully. It is required to determine the poſition of the beams when they are in equilibrio; the pully and beams being in the ſame vertical plane?

V. QUESTION 215, *by* HYPATIA.

From A one end of AB, the diameter of a circle, draw any chord AC, and draw CD perpendicular to AB. Biſect the arch AC in E, and draw BE, meeting CD in G, and CA in F. The lines BG, GF ſhall be equal. Required the demonſtration?

VI. QUESTION 216, *by Mr.* WALLACE, *R. M. College.*

Let a circle be given by poſition and let A and B be two given points; a point C may be found, ſuch, that if any ſtraight line be drawn through it, meeting the circle in D and E and AD, BD, alſo AE, BE be joined, there is a certain given ſpace which is a mean proportional between $AD^2 + BD^2$ and $AE^2 + BE^2$. Required the demonſtration?

VII. QUESTION 217, *by Mr.* LOWRY.

An ellipſe, and a ſtraight line, being given by poſition; a point may be found ſuch, that if any ſtraight line be drawn through that point to meet the ellipſe in two points, and perpendiculars be drawn from thoſe points to the ſtraight line given by poſition, the rectangle of thoſe perpendiculars ſhall always be equal to a certain given ſpace?

VIII. QUESTION 218, *by Mr.* LOWRY.

If the vertical angle and ſum of the ſides of a plane triangle remain conſtant, what is the equation of the curve to which the baſe is always a tangent?

IX. QUESTION 219, *by Mr.* WALLACE.

Let AB, AC be two ſtraight lines given by poſition, and P a given point: it is required to draw a line through P to meet the given lines in D and E, ſo that AD × AE may be to DP × PE in a given ratio.

X. QUESTION 230, *by Mr.* Lowry.

If through a given point P on the ſurface of a ſphere two great circles be drawn to interſect a ſmall circle given by poſition in the points A, B and C, D; and the points AC, BD be joined by great circles which interſect in Q. What is the locus of the point Q?

XI. QUESTION 221, *by Mr.* Lowry.

Find the perſpective repreſentation of a given ſpheroid placed on a horizontal plane; the relative poſitions of the eye, the vertical plane of the picture, and the ſpheroid being given?

XII. QUESTION 222, *by Mr.* Bazley.

It is required to determine the locus of the vertex of a plane triangle, whereof the radius of the inſcribed circle, and difference of the angles at the baſe are given; the middle of the baſe being a fixed point in a ſtraight line given by poſition?

XIII. QUESTION 223, *by* ———.

Let a circle be given by poſition, and let A and B be two given points, a point C may be found, ſuch, that if any ſtraight line be drawn through C to meet the circle in D and E, and AD, BD, AE, BE be joined, $AD^2 + DB^2 : AE^2 + EB^2 :: DC : CE$. Required the demonſtration?

XIV. QUESTION 224, *by Mr.* Ivory.

Suppoſing x to denote any fraction leſs than unit, it is required to aſſign the value of the infinite product $(1 + x)(1 + x^2)(1 + x^4)(1 + x^8)$ &c.?

XV. QUESTION 225, *by Mr.* Cunliffe.

To find integer values for the ſides of a trapezium that may be inſcribed in a circle, ſuch, that its diagonals, diameter of the circumſcribing circle, and area, may all be expreſſed by whole numbers?

XVI. QUESTION 226, *by Mr.* Cunliffe.

Suppoſe a heavy hollow cylinder, of an uniform thickneſs, to roll freely along an inclined plane of a given length and inclination: having given the external diameter of the cylinder together with the time of rolling along the inclined plane, it is required from thence to determine the internal diameter of the cylinder?

XVII. QUESTION 227, *by Mr.* Cunliffe.

Required the ſum of the infinite ſeries

$$\frac{1}{1 \cdot 3^2} + \frac{2}{3 \cdot 5^2} + \frac{3}{5 \cdot 7^2} + \frac{4}{7 \cdot 9^2} + \frac{5}{9 \cdot 11^2} + \text{\&c.}$$

XVIII. QUESTION 228, *by Mr.* Wallace.

Find the nature of a curve, ſuch, that any normal ſhall have a given ratio to the ſegment of the axis intercepted between it and a given point?

XIX. QUESTION 229, *by Mr.* Lowry.

Two equal bodies, connected by a flexible ſtring, are placed on a ſmooth horizontal table, the ſtring being ſtretched out in a direction parallel to the edge of the table. To the middle of this ſtring, there is faſtened another ſtring, at the extremity of which is a given weight that hangs over the edge of the table, in a vertical direction, and, by its deſcent, communicates motion to the other bodies. Now, ſuppoſing the weights of the bodies and the length of the connecting ſtring to be given; it is required to define the motion of the deſcending weight, and to aſcertain its place at the end of any aſſigned time?

XX. QUESTION 220, *by Mr.* LOWRY.

A circle and a ſphere being given by poſition, it is required to determine the perſpective repreſentation of the circle on the ſurface of the ſphere, and to find the rectification of the curve; ſuppoſing the eye to be at the centre of the ſphere?

XXI. QUESTION 231, *by Mr.* LOWRY.

If from two given points on the ſurface of a ſphere, two great circles be drawn to any point D in the circumference of a ſmall circle given by poſition, meeting it again in E and F; then the great circle joining the points E, F will either paſs through a given point P; or a great circle drawn through E and P will cut off from the ſmall circle EDF an arch FG of a given length?

XXII. QUESTION 232, *by Mr.* WALLACE.

Let F be the focus of a conic ſection, and P a given point in the axis. Draw FA to any point of the curve; take PB in the axis equal to FA and join AB. There is a ſtraight line given by poſition which divides AB into parts having to each other a given ratio. Required the demonſtration?

XXIII. QUESTION 233.

In the *Traités de Calcul Différentiel et de Calcul Intégral* of Boſſut, the following theorem is given (but without demonſtration) from the works of John Bernoulli.

"Let A4B (fig. 95, pl. 4.) be the ſemi-circumference of a circle, of which AB, the diameter, is equal to the two ſemi-axes AC, BC of a given ellipſe. Divide the ſemi-circumference A4B into a number, 2, 4, 8, 16, &c. of equal parts, draw ſtraight lines from the point C to all the points of diviſion: take the arithmetical mean of all the ſtraight lines drawn to the points of diviſion denoted by odd numbers, and alſo the arithmetical mean of the ſum of all the lines drawn to the points of the even numbers, and the half of AB; theſe two arithmetic means ſhall be the radii of two circles, the circumference of one of which ſhall be leſs than that of the ellipſe, and the circumference of the other greater." Required the demonſtration?

XXIV. QUESTION 234, *by Mr.* IVORY.

A point may be found within the baſe of every oblique cone, ſuch, that if any plane be drawn through the vertex and the point found to cut the cone, then the vertical angle of the triangle, which is the common ſection of the plane and the cone, will be biſected by the right line drawn from the point found to the vertex of the cone?

XXV. QUESTION 235, *by Mr.* IVORY.

It is required to ſhew that the ſurface of a right cone on an elliptical baſe may be found by the rectification of the ellipſe.

By a right cone on an elliptical baſe is meant ſuch a cone as has an ellipſe for its baſe, and likewiſe the right line, drawn from the vertex to the centre of the ellipſe, perpendicular to the plane of the baſe?

XXVI. QUESTION 236, *by Mr.* IVORY.

From a point aſſumed without a cone, let two tangents be drawn to the ſame conic ſurface, or to the oppoſite and let a plane be drawn through the vertex of the cone and the two points of contact of the tangents: then, if a right line be drawn from the aſſumed point to cut the plane, and either of the conic ſurfaces in two points, that right line will be harmonically divided?

N. B. The demonſtration of ſome properties of all the conic ſections readily follows from this propoſition

XXVII. QUESTION 237, *by Mr.* IVORY.

It is required to determine the poſition of the axes of the repreſentation of a circle given by poſition in a horizontal plane: ſuppoſing the place of the eye to be given, and the plane of the picture perpendicular to the plane of the horizon?

XXVIII. QUESTION 238, *by Mr.* IVORY.

Let n be any whole poſitive number, and put

$Q = \frac{2 \cdot 4 \cdot 6 \cdot 8 \ldots\ldots 2n}{3 \cdot 5 \cdot 7 \cdot 9 \ldots\ldots (2n+1)}$: then the whole value of the triple fluent $\int \frac{dx}{\sqrt{(1-x^2)}} \int \frac{dx}{\sqrt{(1-x^2)}} \int \frac{x^{2n+1}dx}{\sqrt{(1-x^2)}}$, which is generated while x increaſes from 0 to 1, is equal to

$$Q \times \left\{ \frac{1}{(2n+3)^2} + \frac{1}{(2n+5)^2} + \frac{1}{(2n+7)^2} + \frac{1}{(2n+9)^2} + \&c. \right\}?$$

N. B. Obſerve that the caſe $n = 0$ is included by taking $Q = 1$.

XXIX. QUESTION 239, *by Mr.* IVORY.

Let n be a whole poſitive number and put

$P = \frac{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \cdot 8 \ldots 2n}$: then the whole value of the triple fluent $\int \frac{dx}{\sqrt{(1-x^2)}} \int \frac{dx}{\sqrt{(1-x^2)}} \int \frac{x^{2n}dx}{\sqrt{(1-x^2)}}$, which is generated while x increaſes from 0 to 1, is equal to

$$P \times \left\{ \frac{1}{(2n+2)^2} + \frac{1}{(2n+4)^2} + \frac{1}{(2n+6)^2} + \frac{1}{(2n+8)^2} + \&c. \right\} \times \frac{\pi}{2}$$

where π is the ſemi-circumference to the radius 1?

N. B. Obſerve that the caſe $n = 0$ is included by taking $P = 1$.

XXX. PRIZE QUESTION, 240, *by Mr.* IVORY.

Let a plane be drawn through the axis of an oblique cone, ſo as to be perpendicular to the plane of the baſe; and let a right line be drawn to biſect the vertical angle of the triangle which is the common ſection of the plane and the cone: then, if a ſecond plane be drawn through this line (*viz.* the line which biſects the vertical angle of the triangle) to cut the cone, the difference of two portions of the conic ſurface between the two planes, and comprehended in oppoſite angles, may be determined by means of a circular area and a rectilineal ſpace?

Solutions to theſe queſtions muſt come to hand (poſt-paid) by the Firſt of May 1808.

ERRATA.

In page 4, vol. 2, part 2, line 11 from the bottom

for $A \times \frac{x^2u^2}{y^2} + D \times \frac{x^2u^2}{z^2}$ *read* $A \times \frac{x^2u^2}{y^2} + Bu^2 + D \times \frac{x^2u^2}{z^2}$

NEW SERIES

OF THE

MATHEMATICAL REPOSITORY.

VOL. II. PART I.

MATHEMATICAL QUESTIONS.

ARTICLE I.

Solutions to Queſtions propoſed in Number IV.

I. QUESTION 121, *by* SCOTICUS.

If α and β denote given arches, and

$$\text{ſec.}\,\alpha\ \text{ſec.}\,\varphi + \text{tan.}\,\alpha\ \text{tan.}\,\varphi = \text{ſec.}\,\beta.$$

Required the values of ſec. φ and tan. φ?

FIRST SOLUTION, *by* SCOTICUS, *the Propoſer.*

Becauſe

$$\text{ſec.}\,\alpha + \text{tan.}\,\alpha = \frac{1}{\text{ſec.}\,\alpha - \text{tan.}\,\alpha}$$

$$\text{ſec.}\,\varphi + \text{tan.}\,\varphi = \frac{1}{\text{ſec.}\,\varphi - \text{tan.}\,\varphi}.$$

By taking the product of theſe two equations and ſubſtituting ſec. β for ſec. α ſec. φ + tan. α tan. φ we have

$$\text{ſec.}\,\beta + \text{ſec.}\,\alpha\ \text{tan.}\,\varphi + \text{tan.}\,\alpha\ \text{ſec.}\,\varphi$$

$$= \frac{1}{\text{ſec.}\,\beta - (\text{ſec.}\,\alpha\ \text{tan.}\,\varphi + \text{tan.}\,\alpha\ \text{ſec.}\,\varphi)}.$$

But $\text{ſec.}\,\beta + \text{tan.}\,\beta = \dfrac{1}{\text{ſec.}\,\beta - \text{tan.}\,\beta}$ therefore, by comparing this equation with the preceding one, it is manifeſt that

$$\text{ſec.}\,\alpha\ \text{tan.}\,\varphi + \text{tan.}\,\alpha\ \text{ſec.}\,\varphi = \text{tan.}\,\beta.$$

By taking the ſum and difference of this equation, and the equation

$$\text{ſec.}\,\alpha\ \text{ſec.}\,\phi + \text{tan.}\,\alpha\ \text{tan.}\,\phi = \text{ſec.}\,\beta,$$

and reſolving the reſult into factors, we find

$$(\text{ſec.}\,\alpha + \text{tan.}\,\alpha)\,(\text{ſec.}\,\phi + \text{tan.}\,\phi) = \text{ſec.}\,\beta + \text{tan.}\,\beta$$
$$(\text{ſec.}\,\alpha - \text{tan.}\,\alpha)\,(\text{ſec.}\,\phi - \text{tan.}\,\phi) = \text{ſec.}\,\beta - \text{tan.}\,\beta.$$

Hence, $$\text{ſec.}\,\phi + \text{tan.}\,\phi = \frac{\text{ſec.}\,\beta + \text{tan.}\,\beta}{\text{ſec.}\,\alpha + \text{tan.}\,\alpha};$$

$$\text{ſec.}\,\phi - \text{tan.}\,\phi = \frac{\text{ſec.}\,\beta - \text{tan.}\,\beta}{\text{ſec.}\,\alpha - \text{tan.}\,\alpha};$$

and $$\text{ſec.}\,\phi = \text{ſec.}\,\alpha\ \text{ſec.}\,\beta - \text{tan.}\,\alpha\ \text{tan.}\,\beta;$$
$$\text{tan.}\,\phi = \text{ſec.}\,\alpha\ \text{tan.}\,\beta - \text{tan.}\,\alpha\ \text{ſec.}\,\beta.$$

SECOND SOLUTION, *by Mr.* JOHN DAWES.

Put $a = \text{tan.}\,\alpha$, $b = \text{ſec.}\,\beta$, and $x = \text{tan.}\,\phi$, radius 1: then $\surd(1 + a^2) = \text{ſec.}\,\alpha$, $\surd(b^2 - 1) = \text{tan.}\,\beta$, and $\surd(1 + x^2) = \text{ſec.}\,\phi$. Theſe values being written in the propoſed expreſſion, it becomes

$$\surd(1 + a^2)\,\surd(1 + x^2) + ax = b,$$

and by tranſpoſition

$$\surd(1 + a^2)\,\surd(1 + x^2) = b - ax; \quad \ldots\ldots\ldots\ldots(q)$$

ſquaring both ſides

$$(1 + a^2)\,(1 + x^2) = (b - ax)^2;$$

that is, $1 + a^2 + x^2 + a^2x^2 = b^2 - 2abx + a^2x^2$

whence $x^2 + 2abx = b^2 - (1 + a)^2,$

and $x = \sqrt{(1 + a^2)\,(b^2 - 1)} - ab;$

that is, $\text{tan.}\,\phi = \text{ſec.}\,\alpha\ \text{tan.}\,\beta - \text{ſec.}\,\beta\ \text{tan.}\,\alpha.$

Again from the equation q

$$\surd(1+a^2)\surd(1+x^2) = b - ax = b + ba^2 - a\sqrt{(1 + a^2)\,(b^2 - 1)}$$

and dividing by $\surd(1 + a^2)$,

$$\surd(1 + x^2) = b\,\surd(1 + a^2) - a\,\surd(b^2 - 1);$$

that is, $\text{ſec.}\,\phi = \text{ſec.}\,\beta\ \text{ſec.}\,\alpha - \text{tan.}\,\beta\ \text{tan.}\,\alpha.$

Q. E. I.

COR. The ſubſequent equations are eaſily obtained from the preceding, *viz.*

$$\text{ſec.}\,\alpha = \text{ſec.}\,\beta\ \text{ſec.}\,\phi \pm \text{tan.}\,\beta\ \text{tan.}\,\phi.$$
$$\text{tan.}\,\alpha = \text{ſec.}\,\beta\ \text{tan.}\,\phi \pm \text{ſec.}\,\phi\ \text{tan.}\,\beta.$$
$$\text{tan.}\,\beta = \text{ſec.}\,\alpha\ \text{tan.}\,\phi \pm \text{ſec.}\,\phi\ \text{tan.}\,\alpha.$$

Ingenious Solutions to this Queſtion were alſo received from Meſſrs. Amicus, Rev. J. Toplis, Johnſon, *and* J. W. of Leeds.

II. QUESTION 122, *by* AMICUS.

If two tangents PE, PF be drawn to a conic ſection, and the line EF joining the points of contact be produced to meet the directrix in G; then if PI be drawn through the focus to meet EF in I, EI will be to IF as EG to FG, required the demonſtration?

SOLUTION, *by Mr.* JOHN JOHNSON, *Birmingham.*

Fig. 1, pl. 1. Let f be the focus; join Ef, Ff, and draw EH, FK perpendicular to the directrix.

It is a property of the conic ſections, now pretty generally known, that the angle EfP is equal to the angle FfP, and therefore the angle EfI is equal to the angle FfI, fI biſects the angle EfF; wherefore Ef : fF :: EI : FI. Again by the property of the directrix Ef : fF :: EH : FK :: EG : FG, and comparing this proportion with the former we have EI : IF :: EG : FG. *Q. E. D.*

III. QUESTION 123, *by Mr.* JAMES WHALLEY, *Bolton.*

To conſtruct a trapezium there is given one of the ſides, both the diagonals and the angle formed by their interſection, and the angle made by either diagonal and ſide oppoſite to the given one?

FIRST SOLUTION, *by Mr.* W. WALLACE, *R. M. College.*

Suppoſe ABCD (fig. 2, pl. 1.) to be the quadrilateral required, then by hypotheſis AB one of its ſides, and AC, BD, its two diagonals are given, alſo the triangle DEC, formed by the diagonals, and DC the oppoſite ſide, is given in ſpecies.

Produce AB, CD to meet in G, and draw AH parallel to the diagonal BD, then BG : GA :: BD : AH, but the ratio of BD to AH is compounded of the ratios of BD to AC and of AC to AH, that is of BD to AC and of EC to ED, both of which ratios are given ſince BD and AC are given lines and the triangle EDC is given in ſpecies, therefore the ratio of BG to GA is given, and AB being given, AG and BG are both given. Now in the triangle ACG the baſe AG, the vertical angle ACG and AC one of the ſides are known, therefore the triangle mad be eaſily conſtructed, and hence the point C will be known, any this point being determined all the reſt may be found of courſe.

SECOND SOLUTION, *by* AMICUS.

Fig. 2, pl. 1. Take AC equal to one of the given diagonals, and draw AH, CH to make the angles CAH, ACH, equal to the angles made by the interſection of the diagonals and the other given angle reſpectively; produce AH, if neceſſary, till AF be equal to the other given diagonal, and from F to the line CH apply FD equal to the given ſide; complete the parallelogram FDBA and join BC; then ABCD is the trapezium required.

For ſince ABDF is a parallelogram, AB is equal to FD the given ſide, and BD is equal to AF one of the given diagonals, alſo AC is equal to the other diagonal; and the angles ACD, DEC (or HAC, becauſe DE and HA are parallel) are equal to the given ones.

THIRD SOLUTION, *by Mr.* CUNLIFFE, *R. M. College.*

Fig. 2, pl. 1. Let CDBA repreſent the required trapezium, AB being the given ſide; draw BG′ parallel to AC and produce DC to meet it in G′; alſo draw AS parallel to DG′. Then in the triangle DBG′ the angles BDG′ and DBG′ are both given together with the ſide or diagonal DB, whence the triangle itſelf becomes entirely known.

The figure CASG′ is a parallelogram by the conſtruction, therefore G′S = AC a given line, conſequently S is a given point, and SA a right line given by poſition: whence the following conſtruction is pretty obvious.

Having drawn SA, from B apply BA thereto equal to the given ſide, draw AC parallel to BG′ meeting DG′ in C, join CB, DA, and DCBA is the required trapezium, as is evident from the analyſis.

This Queſtion was likewiſe anſwered by Meſſrs. Bazley, Johnſon, *and* Toplis.

IV. QUESTION 124, *by* SENEX.

Let there be two given points in the circumference of a circle given in magnitude and poſition: a circle may be found, given alſo in magnitude and poſition, ſuch, that if from any point in its circumference ſtraight lines be drawn to the given points, and a tangent to the given circle; the ſquare of the tangent ſhall have to the rectangle of the lines drawn to the given points a given ratio.

SOLUTION,

SOLUTION, *by Mr.* BAZLEY.

Let A and B (fig. 3, pl. 1.) be the given points, ABED the given circle, C one of the points in the periphery of the circle which is to be found; draw the tangent CD and join AC, BC, and AB: let E be the point where AC cuts the given circle, and join BE.

Then by hypotheſis, the ratio of $AC \times CB : CD^2 = AC \times CE$ is given; that is, the ratio of CB to CE is given; but ſince the circle ABED and the chord AB are given, therefore the angle AEB is given, and its ſupplement BEC; conſequently the triangle BEC is given in ſpecies, and therefore the angle ACB ſtanding on the given line AB is given. Wherefore, the point C is placed in the circumference of a circle deſcribed on the given line AB, to contain the given angle ACB.

Meſſrs. Amicus, Cunliffe, Johnſon *and* I. W. of Leeds *favoured us with Solutions to this Queſtion.*

V. QUESTION 125, *by Mr.* JOSEPH BREWER, *Preſton.*

Two right lines being given by poſition, a circle may be found, ſuch, that if another circle be deſcribed upon any radius thereof as a diameter, the chord of the arc of this latter circle intercepted by the lines given by poſition ſhall be of a given length.

SOLUTION, *by Mr.* BAZLEY.

Suppoſe AB, AC (fig. 4, pl. 1.) to be the two ſtraight lines given by poſition, apply DE of the given length perpendicular to AB and meeting AC in E; with the centre A and diſtance AE deſcribe the circle ER, which is that which was to be found. For, on AE as a diameter, deſcribe a circle, which will paſs through D, becauſe the angle D is a right one. Draw any other radius AR of the circle RE, whereon, as a diameter, deſcribe the circle APQR, meeting AB in Q and AC in P, and join PQ. Then in the equal circles ADE, APQR, the chords ED, PQ, ſubtending the ſame angle BAC, are equal; that is, PQ = DE = the given length.

Meſſrs. Johnſon *and* J. W. *of* Leeds *alſo ſent anſwers to this Queſtion.*

VI. QUESTION 126, *by Mr.* J. JOHNSON.

Given the poſition of the centres of gravity of the three triangles, formed by lines drawn from the angular points of the triangle to the centre of the inſcribed circle, to determine the poſition of the triangle?

First Solution, *by* Mr. Bazley.

Let ABC (fig. 5, pl. 1.) be the required triangle; P, Q, R, the given points, and O the centre of the inſcribed circle. Draw OPS meeting AB in S; then it is well known that S is the middle of AB and SP = $\frac{1}{3}$SO; through P draw DPE parallel to AB meeting OA, OB in D and E: then AD = $\frac{1}{3}$AO, and if DF be drawn parallel to AC meeting OC in F, CF is = $\frac{1}{3}$CO, and DF paſſes through Q. Again, join FE, whic paſſes through R, becauſe CF = $\frac{1}{3}$CO and BE = $\frac{1}{3}$BO; moreover P, Q, R, are evidently the middle points of DE, DF, FE. Hence DPRQ, EPQR and FQPR are parallelograms; and therefore we have this

Construction.

Having joined P, Q, R, draw DPE parallel to QR taking PD, PE each equal to QR, and draw DQF, ERF; alſo draw DO, EO, biſecting the angles D and E and meeting in O; produce OD to A ſo that OA : OD :: 3 : 2, and in the ſame ratio produce OE to B; laſtly draw AC, BC (meeting in C) parallel to DF, EF, and the thing is done: the demonſtration is plain from the analyſis.

Second Solution, *by the Rev.* J. Toplis.

Let ABC (fig. 6, pl. 1.) be the triangle whoſe poſition is required, AOB, AOC, and BOC the three triangles formed by lines drawn from the angular points to the centre of the inſcribed circle, and G, H, F their centres of gravity. Join GH, HF, FG, and let the lines AGE, BFD, CHP be drawn through the points G, F, H, meeting the lines OB, OC, OA, and (by the property of the centre of gravity) biſecting them in E, D, and P; then it is evident that the lines joining the points A, D; B, P; C, E, will paſs through the points H, G, F reſpectively; and by a well known property of the centre of gravity, BP is equal to 3GP, PC to 3PH; therefore GH and BC are parallel, and BC equal to 3GH. In a ſimilar manner it may be proved that AC is parallel to GF and AB to FH and that AC is equal to 3GF and AB to 3FH; wherefore the triangle ACB is ſimilar to, and equal to thrice the given triangle FGH. Let Q be the centre of the circle inſcribed in the triangle FHG; biſect the lines GQ, HQ in K and I, and draw GI, HK, FK, FI; then it is obvious that the triangle GHQ is ſimilar to the triangle BCO; and ſince GQ, CO, are biſected in K and D, and BC equal to 3GH, it is evident that BD (3FD) is equal to 3HK, that is, FD is equal to HK; and in a ſimilar way it is ſhewn that FE is equal to GI, DH to FK, and EG to FI, wherefore HK, GI, FK, FI are given lines; hence if FD, DH, FE, GE be taken equal to the given lines, the points D, E, will be determined, and if DH, DF, EF be produced till DA is equal to 3DH, BD to 3DF, and EC to 3EF, the poſition of the angular points of the triangle become known.

An ingenious Solution was received from Mr. Johnſon, *the Propoſer.*

VII. QUESTION 127, *by* HYPATIA.

Suppoſe $u = \dfrac{a^{a-n} - x^{x-n}}{a-x}$, what is the value of u when $x = a$?

SOLUTION, *by the Rev.* J. TOPLIS.

The fluxion of $a^{a-n} - x^{x-n}$ is $-$ log. $x \times x^{x-n}\dot{x}$ $- \dfrac{x^{x-n}(x-n)\dot{x}}{x}$, and the fluxion of $a - x$ is $-\dot{x}$; therefore the fluxion of the numerator divided by the fluxion of the denominator is $x^{x-n} \times$ log. $x + x^{x-n} - n\,x^{x-n-1} =$ (when x becomes a) $a^{a-n} \times \left(\text{log.}\,a + 1 - \dfrac{n}{a}\right)$.

The values of fractions of the form $\frac{0}{0}$ are very amply treated upon in the firſt vol. of the Traité du Calcul Différential et Integral par S. F. Lacroix, page 241.

VIII. QUESTION 128, *by* URSA MINOR.

Let AB, AC, be two ſtraight lines given by poſition, and let a ſtraight line DE meet them in D and E, ſo that AD + AE is equal to a given line, and let DE be divided at V, ſo that DV may have to VE a given ratio. Required the locus of the point V?

FIRST SOLUTION, *by* AMICUS.

Analyſis. In fig. 7, pl. 1, take AH = the given ſum and make HI to IA in the given ratio of DV to VE; draw IV to

met

meet AC in F, and let PV, QV be drawn parallel to the lines AC, AB, reſpectively; then becauſe AD + AE is = AH, DH is = AE, and by ſimilar triangles AP : AD :: (VE : DE) AI : AH, and by permutation and diviſion IP : AI :: HD = AE : AH; and again by ſimilar triangles and permutation AQ = PV : IH :: AE : AH; therefore IP : PV :: (IA : AF) IA : IH which is a given ratio, and ſhews that AF is = to IH, therefore the locus of the point V is the ſtraight line IF given by poſition. Hence this conſtruction. Divide the given ſum AH into two parts AI, HI, having the given ratio, make AF = IH and the line joining the points I, F, is the locus required.

It is evident from hence that if two bodies whoſe weights are proportional to the ſegments VE, VD, be placed on the inclined planes AB, AC, and connected by a ſtring whoſe length is AH and which paſſes over the common ſection of the planes at A, the locus of the centre of gravity of the two bodies will be the ſtraight line IF.

If the difference, inſtead of the ſum of AD, AE, had been given the locus would alſo be a ſtraight line given by poſition; for if *h*A be taken equal to the given difference, the analyſis and conſtruction will be preciſely the ſame as above.

But the problem may be made more general, for if R and S be any given lines and the ſum or difference of the rectangles AD × R, AE × S be equal to a given ſpace, then the locus of the point V will ſtill be a ſtraight line.

For let AK × R be equal to the given ſpace, and divide AK at I ſo that AI may be to IK in the given ratio of VE to DV; then becauſe AD × R + AE × S = AK × R, we have A × S = DK × R. And by ſimilar triangles &c. as in t preceding Analyſis,

IP : AI :: KD : AK :: KD × R : AK × R;
also AQ = PV : KI :: AE : AK :: AE × S : AK × S;
therefore IP : PV :: (AI : IF) :: AI × S : KI × R, a given ratio, and it is obvious that AF is = to KI therefore the locus of the point V is a ſtraight line IF as before.

The ſame Algebraically.

Draw VP parallel to AC and VQ to AB. Put s = the given ſum of AD, AE; x = AP = VQ, y = PV = AQ, and $m : n$ the given ratio of DV to VE, then by ſimilar triangles

$$n : m :: x : \frac{mx}{n} = DP,$$

$$m : n :: y : \frac{ny}{m} = QE,$$

hence $x + \frac{mx}{n} = \frac{m+n}{n} . x =$ AD,

and $y + \frac{ny}{m} = \frac{m+n}{m} . y =$ AE;

therefore $\frac{m+n}{n} . x + \frac{m+n}{m} . y = s$, an equation to a ſtraight line.

Now when $y = 0$, x is $= \frac{sn}{m+n}$,

and when $x = 0$, y is $= \frac{sm}{m+n}$. And theſe values of x and y determine the points where the locus meets the ſtraight lines AB, AC: hence if AI be taken $= \frac{sn}{m+n}$ and AF $= \frac{sm}{m+n}$ the ſtraight line IF will be the locus required.

SECOND SOLUTION, *by Mr.* BAYLEY.

(Fig. 8, pl. 1.) Divide the given ſum of AD and AE in the given ratio of DV to VE; take on AC, AI, equal to the firſt part, and on AB take AF, equal to the other part, and join IF, which is the required locus. For, take on AC any diſtance IE, and on FA take FD equal IE (the one towards the point A, the other from it); let DE meet IF in V, and draw DG parallel to AC, to meet IF in G. Then it is plain that AD + AE = AF + AI = the given sum by conſtruction; and ſince the triangles AFI, DFG; DVG, EVI, are ſimilar; therefore
EV : VD :: (IE =) DF : DG :: AF : AI, that is;
EV : VD :: AF : AI, which is the given ratio, by conſtruction.

Solutions were alſo received from Meſſrs. Cunliffe, Johnſon, Toplis, *and* J. W. *of* Leeds.

IX. QUESTION 129, *by Mr.* CUNLIFFE, *R. M. College.*

To find two rational numbers ſuch, that if a given number n be ſubtracted from either of them, or from their ſum or difference, the four remainders ſhall all be rational ſquares.

SOLUTION,

SOLUTION, *by* Mr. CUNLIFFE, *the Propoſer.*

Let x^2+n and y^2+n denote the two numbers, by which notation the two firſt conditions of the queſtion will evidently be ſatisfied: it therefore only remains to determine ſuch values of x and y as ſhall make the two remaining conditions squares, *viz.* x^2+y^2+n and x^2-y^2-n, both ſquares.

Aſſume $x+v$ for the root of the firſt, that is, put $x^2+y^2+n=(x+v)^2=x^2+2xv+v^2$, which gives $x=\frac{y^2+n-v^2}{2v}$, by means of which $x^2-y^2-n=\frac{(y^2+n-v^2)^2}{4v^2}-y^2-n$ $=\frac{y^4-6v^2y^2+2ny^2-6nv^2+v^4+n^2}{4v^2}=$ a ſquare; therefore $y^4-6v^2y^2+2ny^2-6nv^2+v^4+n^2=$ a ſquare; Aſſume $s-y^2$ for the root of the preceding ſquare, that is, put $y^4-6v^2y^2+2ny^2-6nv^2+v^4+n^2=(s-y^2)^2=$ $s^2-2sy^2+y^4$; from whence $y^2=\frac{s^2-v^4+6nv^2-n^2}{2\times(s+n-3v^2)}=$ a ſquare; take $s=n-v^2$, then $y^2=\frac{s^2-v^4+6nv^2-n^2}{2\times(s+n-3v^2)}=\frac{nv^2}{n-2v^2}$ $=\frac{n^2v^2}{n^2-2nv^2}=$ a ſquare; therefore n^2-2nv^2 muſt be a ſquare.

Put $n^2-2nv^2=(rv-n)^2=r^2v^2-2rvn+n^2$, whence $v=\frac{2rn}{r^2+2n}$; and $y=\frac{nv}{rv-n}=\frac{2rn}{r^2-2n}$: and from what has been deduced $x=\frac{y^2+n-v^2}{2v}=\frac{32r^4n^2+(r^4-4n^2)^2}{4r\times(r^2+2n)\times(r^2-2n)^2}$, where r may be taken at pleaſure.

Example 1. Take $r=1$, and ſuppoſe the given number $n=1$; then $y=\frac{2rn}{r^2-2n}=-2$, and $x=\frac{32r^4n^2+(r^4-4n^2)^2}{4r\times(r^2+2n)\times(r^2-2n)^2}$ $=\frac{41}{12}$; whence $x^2+1=\frac{1825}{144}$ and $y^2+1=5$, which two numbers will anſwer.

Example 2. Take $r = 1$, and $n = 2$, then $y = -\frac{4}{3}$, and $x = \frac{353}{180}$; whence $x^2 + 2 = \frac{189409}{32400}$; and $y^2 + 2 = \frac{34}{9} = \frac{34 \times 3600}{9 \times 3600} = \frac{122400}{32400}$.

X. QUESTION 130, *by* TRIGONOMETRICUS.

It is required to demonſtrate that the tangent of an arch of 9° is equal to $1 + \sqrt{5} - \sqrt{(5 + 2\sqrt{5})}$, the radius being 1?

FIRST SOLUTION, *by Mr.* J. JOHNSON.

Put s for the ſine and c for the coſine of 18°; then (Emerſon's Trigonometry, page 12,) the tangent of half the arc, or of 9° is $= \sqrt{\frac{1-c}{1+c}} = \sqrt{\frac{(1-c)^2}{1-c^2}} = \frac{1-c}{s}$. But it is well known that the ſine of 18° is $= \frac{\sqrt{5}-1}{4}$, therefore its coſine is $= \sqrt{(1-s^2)} = \frac{1}{4}\sqrt{(10+2\sqrt{5})}$, and $\frac{1-c}{s} = \left\{ 1 - \frac{1}{4}\sqrt{(10+2\sqrt{5})} \right\} \div \frac{\sqrt{5}-1}{4} = \frac{4-\sqrt{(10+2\sqrt{5})}}{\sqrt{5}-1} = 1 + \sqrt{5} - \sqrt{(5+2\sqrt{5})}$ = tangent of 9° as required.

SECOND SOLUTION, *by Mr.* CUNLIFFE.

Let t denote the tangent of an arch A, radius 1; then the writers on trigonometry have ſhewn that the tangent of 5A $= \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$; and this formula will alſo expreſs the tangents of the following arches *viz.* 5A ± 180 and 5A ± 2 × 180.

Suppoſe 5A = 45°, then the tangent of $5A = 1 = \frac{5t-10t^3+t^5}{1-10t^2+5t^4}$, whence $t^5 - 5t^4 - 10t^3 + 10t^2 + 5t - 1 = 0$, the five roots of which equation are the tangents of 9°, 45°, −27°, 81° and −63°. And, ſince the tangent of 45° is a root of the above equation, therefore $t - 1$ muſt be a diviſor thereof, and the quotient thence ariſing will be $t^4 - 4t^3 - 14t^2 - 4t + 1 = 0$, completing the ſquare by adding $20t^2$ to both ſides $t^4 - 4t^3 + 6t^2 - 4t + 1 = 20t^2 = 4t^2 \times 5$; taking the roots $t^2 - 2t + 1 = \pm 2t\sqrt{5}$, or $t^2 - 2t \times (1 \pm \sqrt{5}) = -1$; again completing the ſquare $t^2 - 2t \times (1 \pm \sqrt{5}) + (1 \pm \sqrt{5})^2 = 5 \pm 2\sqrt{5}$; taking the roots, &c. $t = 1 \pm \sqrt{5} \mp \sqrt{(5 \pm 2\sqrt{5})}$, and the two poſitive values of t are $1 + \sqrt{5} - \sqrt{(5 + 2\sqrt{5})}$ and $1 + \sqrt{5} + \sqrt{(5 + 2\sqrt{5})}$, the former of which muſt therefore be the tangent of 9°, and the latter the tangent of 81°.

Ingenious Solutions were alſo received from Meſſrs. Amicus, Dawes, *and* J. W. *of* Leeds.

The next Six Queſtions are ſelected from the Cambridge Univerſity Calendars for 1804 *and* 1805.

XI. QUESTION 131, *from the* CAMB. UN. CAL.

A given parabolic conoid is put into a fluid, and when it is in a poſition of permanent equilibrium the baſe is wholly extant, and inclined to the horizon; having given its ſpecific gravity, relatively to that of the fluid, it is required to determine the dimenſions of the part immerſed?

FIRST SOLUTION, *by Mr.* LOWRY.

LEMMA. A PROBLEM. Fig. 9. Pl. 1.

To find the centre of gravity of an oblique ſegment of a parabolic conoid.

It is demonſtrated at prop. 1, ſect. 4, part 3, of Dr. Hutton's Menſuration, that the ſection made by a plane cutting a paraboloid oblique to the axis will be always an ellipſe, and that all the parallel ſections will be like and ſimilar figures. Hence, if AB be the tranſverſe axis of the elliptic baſe, and LR a diameter to AB, the ſections parallel to the baſe will be ſimilar to the ſection ACB and have their centres in the diameter LR; and therefore the centre of gravity of the ſolid will alſo be in that diameter. Put a = the baſe ACB, b = the diameter LR, s = the ſine of the angle LRB, x = any variable abſciſſa LD, whoſe ordinate is ab; o the centre of gravity of the ſegment aLb and O that of ALB. Then $b : x :: a : \frac{ax}{b}$ = the area of the ſection at D, and $\frac{asx\dot{x}}{b}$ will be the fluxion of the ſolid aLb; and therefore, by the common formula for finding the centre of gravity

$$Lo = \frac{\int \frac{as}{b} x^2 \dot{x}}{\int \frac{as}{b} x\dot{x}} = \frac{\int x^2 \dot{x}}{\int x\dot{x}} = \frac{2x^3}{3x^2} = \frac{2}{3} x.$$

When x is equal to b, then Lo is equal to $\frac{2}{3} b$.

Now let ABC (fig. 10. pl. 1.) be a ſection of the conoid made by a plane paſſing through the axis HC, and through the point A in the baſe which is neareſt to the ſurface of the fluid. Let EF be the line where the plane of the fluid meets the ſection, and EFC the part of the body immerſed in the fluid; then, if G be the centre of gravity of the whole conoid, and O that of the immerſed part EFC, the body will be in a poſition of permanent equilibrium when the line joining the points G, O, is perpendicular to EF the ſurface of the fluid; and the content of the immerſed part EFC, will be to that of the whole conoid, as the ſpecific gravity of the body to that of the fluid; therefore if s be the content of the conoid, and 1 to m the relative proportion of the ſpecific gravities of the ſolid and fluid, then $\frac{s}{m}$ will be the content of the immerſed part EFC; and, to determine its dimenſions, draw Eb parallel to AB meeting the axis in K, and let PQ, the diameter to the ordinate EF, meet Eb in D, and draw FN and GL perpendicular to Eb and PD; then PQ being parallel to KC, PD will be perpendicular to Eb, and becauſe EP = PF, ED will be = DN and Nb = 2DK.

Put p = the parameter of the axis, and n = the area of a circle whoſe diameter is unity; then the content of the immerſed ſegment EFQ is equal to a cylinder on the circular baſe EN and altitude half the diameter PQ, that is, = to $EN^2 . \frac{PQ}{2} . n$ (pa. 378, cor. 2, of the work referred to in the lemma.) And by Emerſon's conic ſections, prop. 19, $EN^2 = 4ED^2 = PQ . 4p$, or $PQ = \frac{EN^2}{4p}$, therefore the content of the ſegment is $= \frac{EN^4}{4p} . \frac{n}{2} = \frac{s}{m}$ and $EN = \sqrt[4]{\frac{8ps}{nm}}$, a given quantity, for which put $2a$, and then ED will be = to a. And becauſe $PQ . 4p = EN^2$, PQ will be = to $\frac{1}{4p} \sqrt{\left(\frac{8ps}{nm}\right)} = \sqrt{\left(\frac{s}{2pnm}\right)}$, which denote by $3b$. Now, when G is the centre of gravity of the conoid, GC is $= \frac{2}{3}$CH $= c$; and when O is the centre of gravity of the ſegment, OQ is $= \frac{2}{3}$PQ (by the lemma), and OP $= \frac{1}{3}$PQ $= b$.

Let GO meet EF in I, then by Emerſon's Conics, prop. 4.
p : EN :: Nb = 2DK : NF = 2DP, or
p : 2ED :: DK : DP. But the right angled triangles EDP, OPI, OLG being ſimilar, we have
ED : OL :: PD : LG = DK, and ex æquo, p : 2OL :: DK : DK; therefore OL is given $= \frac{1}{2}p$. Draw QT parallel to LG meeting the axis in T; then becauſe OQ is $= 2b$, GT = LQ is given $= \frac{1}{2}p + 2b$, and therefore TC is $=$ GC $-$ GT $= c - \frac{1}{2}p - 2b$; and by the property of the curve QT^2 = (DK^2) $= p \times$ TC $= p\left(c - \frac{1}{2}p - 2b\right)$, therefore DK $= \sqrt{\left(pc - \frac{1}{2}p^2 - 2bp\right)}$, a given quantity; from whence the ordinate EK and the abſciſſa KC become known, and the dimenſions of the immerſed ſegment are then eaſily determined.

The poſition of EF may alſo be found by a very ſimple conſtruction: for having taken GT on the axis CG equal to the given length, and drawn QT perpendicular to GC meeting the curve in Q, let QS and QR be taken on QT produced, each equal to ED, or half EN; then the lines SE, RF being drawn parallel to the axis, will meet the curve in the points E and F required.

SECOND SOLUTION, *by Mr.* CUNLIFFE.

Let ADB (fig. 11, pl. 1.) repreſent a ſection of the given parabolic conoid through its axis CD; alſo let LDH repreſent a ſection of the immerſed part, lying in the ſame plane with the ſection ADB. Draw Lm, Hn and pd parallel to the axis CD; the latter, *viz.* pd biſecting LH in s: alſo draw Hqr and Lt parallel to AB.

Let G, in CD, be the centre of gravity of the whole parabolic conoid ADB; alſo let g, in pd, be the centre of gravity of the immerſed ſegment LDH.

Then it is well known that when the floating body is in a poſition of permanent equilibrium, the right line joining gG will be perpendicular to LH, the ſurface of the fluid. Draw Gv parallel to AB meeting pd in v, and in the laſt mentioned circumſtance the triangles Gvg and sHq will be ſimilar.

Put AC $=$ BC $= b$, CD $= a$, Cn $=$ rH $= y$, Cm $=$ Lt $= u$; then mn $= y + u$, qH $= \frac{1}{2}(y + u)$ and Cp $=$ Gv $= \frac{1}{2}(y - u)$; and by the known property of the parabola,

$b^2 : a :: y^2 : \frac{ay^2}{b^2} =$ rD; and in like manner $\frac{au^2}{b^2} =$ tD: whence rD — tD = rt = $\frac{a}{b^2} \times (y^2 - u^2)$, and qs = ½rt = $\frac{a}{2b^2} \times (y^2 - u^2)$; and by the ſimilar triangles Gvg, sHq, we have qs : qH :: Gv : vg, whence vg = $\frac{\mathrm{qH} \times \mathrm{Gv}}{\mathrm{qs}} = \frac{b^2}{2a}$.

Again, by the known property of the parabola, CB² : CD :: qH² : sd, whence sd = $\frac{\mathrm{CD} \times \mathrm{qH}^2}{\mathrm{CB}^2} = \frac{a \times (y+u)^2}{4b^2}$, and it is well known that sg = ⅓sd = $\frac{a \times (y+u)^2}{12b^2}$:

Hence, Cr = CD — rD = $a - \frac{ay^2}{b^2} = \frac{a \times (b^2 - y^2)}{b^2}$.

$$\mathrm{ps} = \mathrm{Cr} + \mathrm{qs} = \frac{a \times (b^2 - y^2)}{b^2} + \frac{a \times (y^2 - u^2)}{2b^2} = \frac{a}{2b^2} \times \left\{ 2b^2 - (y^2 + u^2) \right\},$$

and
$$\mathrm{pg} = \mathrm{ps} + \mathrm{sg} = \frac{a}{2b^2} \times \left\{ 2b^2 - (y^2 + u^2) \right\} + \frac{a}{12b^2} \times (y+u)^2$$
$$= \frac{a}{12b^2} \times \left\{ 12b^2 - 6(y^2 + u^2) + (y+u)^2 \right\}.$$

But pg = ps + sg = pv + vg = CG + vg, that is,
$$\frac{a}{12b^2} \times \left\{ 12b^2 - 6(y^2 + u^2) + (y+u)^2 \right\} = \frac{a}{3} + \frac{b^2}{2a} = \frac{2a^2 + 3b^2}{6a}.$$

Whence $12b^2 - 6(y^2 + u^2) + (y+u)^2 = \frac{2b^2}{a^2} \times (2a^2 + 3b^2)$ (1).

By Hutton's Menſuration part 3rd. ſect. 6th. 4 × BC² × DC × $\frac{q}{2}$ expreſſes the content of the parabolic conoid ADB; and $\frac{\mathrm{mn}^4 \times \mathrm{DC} \times \frac{1}{2}q}{4 \times \mathrm{BC}^2}$ = the content of the oblique ſegment LHD, where $q = {\cdot}7854$.

Then by the principles of hydroſtatics,
$4 \times \mathrm{BC}^2 \times \mathrm{DC} \times \frac{q}{2} : \frac{\mathrm{mn}^4 \times \mathrm{DC} \times \frac{1}{2}q}{4 \times \mathrm{BC}^2} :: m : n$; *m* and *n* being the ſpecific gravities of the fluid and floating body; whence

$mn^4 = 16 \times BC^4 \times \frac{n}{m}$, or $mn = 2BC\sqrt[4]{\frac{n}{m}}$; that is, $y + u = 2b\sqrt[4]{\frac{n}{m}}$, or $(y+u)^2 = 4b^2\sqrt{\frac{n}{m}}$ .... (2).

Now from the equations 1 and 2 the value of y may be ſound by a quadratic equation.

Cor. From what is deduced in the ſolution, it appears that the centre of gravity of the immerſed ſegment LDH is always at a given perpendicular diſtance from the baſe of the parabolic conoid, whatever are the relative denſities of the fluid and floating body.

For $pg = \frac{a}{3} + \frac{b^2}{2a} = \frac{2a^2 + 3b^2}{6a}$ = a given length.

XII. QUESTION 132, *from the* Camb. Un. Cal.

Find the centre of gravity of a cylindrical portion of the atmoſphere meaſured from the earth's ſurface, the force of gravity being conſtant?

First Solution, *by Mr.* Lowry.

Fig. 12, pl. 1. Let ABC be the baſe, and DC the axis of a cylindrical portion ABGH of the atmoſphere. Imagine this cylinder to be divided into an indefinite number of thin parts, or laminæ of equal thickneſs; then it is evident that the quantity of matter in each of theſe laminæ will be proportional to the denſity of the atmoſphere at the reſpective altitudes. Conceive a ſolid EAFB, ſymetrical to the axis DC, to be deſcribed on the ſame baſe as the cylinder, and compoſed of particles of matter of the ſame denſity as the air at the earth's ſurface and divided into laminæ of the ſame thickneſs as thoſe into which the cylinder is divided: then if the nature of the ſolid be ſuch, that the quantity of matter in each of the laminæ of the ſolid is equal to the quantity of matter in each of the correſponding laminæ of the cylinder, it is evident that the ſolid AEFB will have the ſame centre of gravity S, as the cylinder AGHB. Put a = the area of the baſe, x = the altitude DC, d = the denſity of the air at the earth's ſurface, and y = the denſity at the height DC. Then the quantity of matter in the laminæ GHK is $= ay$, which, by hypotheſis, is = to $d \times$ (area EDF), therefore EDF $= \frac{a}{d} \times y$, and the fluxion of the quantity of matter in the ſolid $= \frac{a}{d} \times y\dot{x}$; and by the common formula for finding the centre of gravity of a ſolid, we have

$$CS = \frac{\int \frac{a}{d} \times yx\dot{x}}{\int \frac{a}{d} \times y\dot{x}} = \frac{\int yx\dot{x}}{\int y\dot{x}}.$$

Now, the relation between the denſity and altitude being expreſſed by the equation $\frac{x}{n} = \log. \frac{d}{y}$, where n is $= 27819$ feet, the height of a homogenious atmoſphere of the ſame denſity as the air at the earth's ſurface, we have $\dot{x} = -n \times \frac{\dot{y}}{y}$, and

$$CS = \frac{\int yx\dot{x}}{\int y\dot{x}} = \frac{\int -nx\dot{y}}{\int -n\dot{y}} = \frac{C - nxy + \int ny\dot{x}}{c - ny} = \frac{C - nxy - n^2y}{c - ny}.$$

But when y is $= d$, x is $= 0$, and then $C - n^2d \quad 0$, or, $C = n^2d$. Alſo $c - nd$ is then $= 0$, or $c = nd$; therefore $CS = \frac{n^2d - nxy - n^2y}{nd - ny} = n - \frac{xy}{d - y}$; which is a general expreſſion for the height of the centre of gravity for any finite altitude.

If we put $d - y = u$, then $\frac{d}{y} = \frac{1}{1 - \frac{u}{d}}$, and the log. $\frac{d}{y} =$

$$\log. \frac{1}{1 - \frac{u}{d}} = \frac{u}{d} + \frac{u^2}{2d^2} + \frac{u^3}{3d^3} + \&c. = \frac{x}{n}, \text{ or}$$

$$x = n \times \left(\frac{u}{d} + \frac{u^2}{2d^2} + \frac{u^3}{3d^3} + \frac{u^4}{4d^4} + \&c.\right) \text{ and therefore}$$

$$\frac{xy}{d - y} = \frac{xy}{n} = n \times y \times \left(\frac{1}{d} + \frac{u}{2d^2} + \frac{u^2}{3d^3} + \frac{u^3}{4d^4} + \&c.\right)$$

$$\text{or } n - \frac{xy}{d - y} = n - n \times y \times \left(\frac{1}{d} + \frac{u}{2d^2} + \frac{u^2}{3d^3} + \frac{u^3}{4d^4} + \&c.\right).$$

Now when y is $= d$, or $u = 0$, all the terms of the ſeries except the firſt will vaniſh, and we have $n - \frac{xy}{d - y} = n - n = 0$, as it ought to be.

Again, when y is $= 0$, or the altitude is infinite, u is $= d$, and we have

$$n - \frac{xy}{d-y} = n - 0 \times n \times (1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \&c.)$$

where it is evident that the product of the ſeries by 0 is = to 0, and therefore when the altitude is infinite $n - \frac{xy}{d-y} = n$.

The problem may alſo be eaſily reſolved by means of the logarithmic curve; for if the line CB (fig. 13, pl. 1.) denote the denſity of the air at the earth's ſurface, and the logarithmic curve BDE whoſe ſubtangent is n, be deſcribed; then the ordinates correſponding to the ſeveral altitudes will denote the denſities of the air at thoſe altitudes reſpectively (ſee Dalby's Courſe of Mathematics, vol. 2d, art. 444): and the centre of gravity of the cylindrical ſpace, whoſe altitude is CD, will evidently be at the ſame diſtance from the ſurface of the earth, that the centre of gravity of the logarithmic ſpace CDAB is from the baſe CB. Put CB $= d$, $x =$ DC, and $y =$ DA; then the fluxion of the ſpace CDAB $= y\dot{x}$, and therefore the diſtance of the centre of gravity of this ſpace from the line CB is $= \frac{\int yx\dot{x}}{\int y\dot{x}}$. Alſo the equation of the curve is $x = n \times \log. \frac{d}{y}$, and therefore the proceſs for finding the centre of gravity of the logarithmic ſpace is preciſely the ſame as before.

If S be the centre of gravity of the ſpace CDAB, and SI, AV, be drawn perpendicular to CB, then we have SI $= n - \frac{x\dot{y}}{d-y} = n - \frac{\mathrm{CD} . \mathrm{DA}}{\mathrm{VB}} = \frac{n . \mathrm{VB} - \mathrm{CD} . \mathrm{DA}}{\mathrm{VB}}$. Or, becauſe n : VB is = to the area of the ſpace CDAB (ibid. art. 443), SI is = to the $\frac{\text{area ABV}}{\mathrm{VB}}$.

Now when CD is infinite, the points C, V, coincide, and the area of the whole logarithmic ſpace, included between the aſymptote and the curve, is $= n . \mathrm{CB}$, therefore in this caſe SI is $= \frac{n . \mathrm{CB}}{\mathrm{CB}} = n$, the ſame as determined above by expanding the logarithmic quantities.

SECOND SOLUTION, *by Mr.* BAZLEY.

If x denote the altitude from the ſurface, and y the denſity of the air at that altitude, then will hyp. log. $\frac{1}{y} = \frac{x}{p}$, where $p = 29725$ feet, the denſity at the ſurface being unity; this is well known. Now if the area of the baſe of the cylinder be taken for brevity's ſake $= 1$, we ſhall have the fluxion of the quantity of matter $= y\dot{x} = -p\dot{y}$ by the equation above, and the fluent is $-py$: but when $y = 1$ this fluent ſhould vaniſh, therefore, by correction, the quantity of matter is $p - py$.

Again, to obtain the fluent of $yx\dot{x} = -px\dot{y}$, aſſume it $= -pxy + m$; whence $-px\dot{y} = -px\dot{y} - py\dot{x} + \dot{m}$: therefore $\dot{m} = py\dot{x} = -p^2\dot{y}$, and $m = -p^2y$; conſequently the fluent of $yx\dot{x}$ is $-pxy - p^2y$, which ought to vaniſh when $x = 0$, or $y = 1$; therefore, by correction, it is $p^2 - p^2y - pxy$. Hence the diſtance of the centre of gravity from the ſurface

$$= \frac{\int yx\dot{x}}{\int y\dot{x}} = \frac{p^2 - p^2y - pxy}{p - py} = p - \frac{xy}{1-y}.$$

Mr. Toplis *alſo ſent a Solution to this Queſtion.*

XIII. QUESTION 133, *from the* CAMB. UN. CAL.

Required the nature of the curve along which a heavy body, deſcending by the force of gravity, will preſs upon the curve at any point, with a force proportional to the ordinate at that point?

FIRST SOLUTION, *by Mr.* CUNLIFFE.

The preſſure of a body in motion againſt any point of a curve is manifeſtly compoſed of two parts, namely, the ſimple ſtatical preſſure and the preſſure ariſing from the centrifugal force, or the force whereby the body endeavours to recede from the centre of curvature at any point: and upon the principle juſt cited it may be eaſily ſhewn that the quadrant of a circle, one of whoſe ſemi-diameters is perpendicular to the horizon will ſatisfy the queſtion.

Let ABS (fig 14, pl. 1.) be the quadrant of a circle having the ſemi-diameter SB perpendicular to the horizon; and ſuppoſe a heavy body C, deſcends from reſt at A, along the arch AC: draw CD perpendicular to AS, alſo draw the radius CS.

Put AS = SB = r, DC = y, v = the velocity of the deſcending body per ſecond at C, and $g = 16\frac{1}{12}$ feet.

Then by known principles $\frac{2gy}{r}$ will be as the ſimple ſtatical preſſure at C; and $\frac{v^2}{r}$ will be as the preſſure ariſing from the centrifugal force; therefore $\frac{2gy}{r} + \frac{v^2}{r}$ will be as the whole preſſure at C: and becauſe the body deſcends from reſt at A, $v = 2\sqrt{gy}$, or $v^2 = 4gy$; whence $\frac{2gy}{r} + \frac{v^2}{r} = \frac{2gy}{r} + \frac{4gy}{r} = \frac{6gy}{r}$: but g and r are both given; wherefore the whole preſſure at C is as the ordinate y = DC.

Cor. The whole preſſure againſt any point C, of the quadrant, is thrice as great as the ſimple ſtatical preſſure at that point.

Second Solution, *by Mr.* Lowry.

Let ACB (fig. 14, pl. 1.) be the curve required, and C the poſition of the body when it has deſcended through the arc AC. Let mCn be a tangent to the curve at C, and CR, (drawn perpendicular to mn) the radius of curvature at that point. Draw the abſciſſa AD parallel to the horizon, and the ordinate DC perpendicular to AD. Then the preſſure of the body upon the curve will be in the direction RC, and will be compoſed of two parts, the one ariſing from the force of gravity, and the other from the actual velocity of the body. Put AD = x; DC = y, the ſpace deſcended through, or the arc AC = s; the velocity of the body at C = v, and the meaſure of the accelerative force of gravity = g.

Then the force of gravity in the direction DC, is to its force in the direction SC, as radius to the coſine of the angle CSD or mCD, that is, as $\dot{s} : \dot{x}$; and therefore the preſſure cauſed by gravity is = $g \times \frac{\dot{x}}{\dot{s}}$. Again ſince the velocity of the body in the curve at C, is the ſame as in the circle whoſe radius is CR, it is evident that the preſſure ariſing from the velocity will be the ſame as the centrifugal force in the circle of curvature at C, that is = $\frac{v^2}{\mathrm{CR}}$. But ſince the body begins to deſcend from A, v^2 is $= 2gy$; therefore the preſſure ariſing from the velocity is $= \frac{2gy}{CR}$, and the whole preſſure upon the curve $= g \times \left(\frac{\dot{x}}{\dot{s}} + \frac{2y}{CR}\right)$. But, by the queſtion, the preſſure is proportional to the ordinate DC, therefore $g\left(\frac{\dot{x}}{\dot{s}} + \frac{2y}{CR}\right) = \frac{m}{n} \cdot y$, where $\frac{m}{n}$ denotes a conſtant ratio.

Now when $\dot{s}$ is conſtant, the radius of curvature is $= \frac{\dot{s}\dot{y}}{\ddot{x}}$, and putting this for CR in the equation above, we have

$$\frac{\dot{x}}{\dot{s}} + \frac{2y\ddot{x}}{\dot{s}\dot{y}} = \frac{m}{ng}\,y;$$

multiply both ſides of this equation by $\frac{\dot{s}\dot{y}}{\sqrt{y}}$,

$$\text{then } \frac{\dot{x}\dot{y}}{\sqrt{y}} + 2\ddot{x}\sqrt{y} = \frac{m}{ng} \cdot \dot{s}\dot{y}\sqrt{y},$$

take the fluents and then

$$2\dot{x}\sqrt{y} = \frac{2m}{3ng} \cdot \dot{s}y^{\frac{3}{2}}, \text{ or } \dot{x} = \frac{m}{3ng} \cdot \dot{s}y.$$

To exterminate $\dot{s}$, we have $\dot{s} = \sqrt{(\dot{x}^2 + \dot{y}^2)}$,

$$\text{therefore } \dot{x} = \frac{m}{3ng} \cdot \sqrt{(\dot{x}^2 + \dot{y}^2)} \cdot y,$$

ſquaring both ſides and reducing we have

$\dot{x} = \frac{my\dot{y}}{\sqrt{(9n^2g^2 - m^2y^2)}}$, and taking the fluents

$\dot{x} = -\sqrt{\left(\frac{9n^2g^2}{m^2} - y^2\right)}$, which gives $x = \frac{3ng}{m}$ when $\dot{y}$ is $= 0$, but x, ſhould then be alſo $= 0$; therefore the correct fluent is $x = \frac{3gn}{m} - \sqrt{\left(\frac{9n^2g^2}{m^2} - y^2\right)}$; and $\frac{6ng}{m} \cdot x - x^2$ is $= y^2$; which is an equation to a circle whoſe radius is $\frac{3ng}{m}$.

Solutions were alſo received from Meſſrs. Bazley *and* Toplis.

XIV.

XIV. QUESTION 134, *from the* CAMB. UN. CAL.

Let the curve APp be generated by taking the ordinate MP always equal to the difference of the chord AC, and the verſed ſine AM, of the circular arc AC. It is required to determine the greateſt ordinate of the curve, and alſo its area?

FIRST SOLUTION, *by the Rev. Mr.* TOPLIS.

Fig. 15, pl. 1. Put AM $= x$, PM $= y$, and the diameter AB $= a$; then, by the property of the circle, AC $= \sqrt{(\mathrm{AM}\ \mathrm{AB})}$ $= \sqrt{(ax)}$, and therefore $y = \sqrt{(ax)} - x$ the equation of the curve.

To find when y is a maximum we have $\frac{\dot{x}\sqrt{a}}{2\sqrt{x}} - \dot{x} = 0$, or $\sqrt{a} = 2\sqrt{x}$, and conſequently $x = \frac{1}{4}a$ and $y = \sqrt{\frac{a^2}{4}} - \frac{1}{4}a$ $= \frac{1}{2}a - \frac{1}{4}a = \frac{1}{4}a$.

Again, to find the area we have $y\dot{x} = \dot{x}\sqrt{(ax)} - x\dot{x}$, and the fluent $= \frac{2}{3}\sqrt{(ax^3)} - \frac{x^2}{2} =$ (when $x = a$) to $\frac{2}{3}a^2 - \frac{a^2}{2}$ $= \frac{1}{6}a^2$ the area required.

The whole might however have been determined without fluxions, for the curve is only a portion of the common parabola, as is evident from the equation obtained above, for $y + x$ is $=$ to $\sqrt{(ax)}$, or $(y + x)^2 = ax = x\sqrt{2} \times a\sqrt{\frac{1}{2}}$, which is an equation to a parabola whoſe ordinate is $y + x$, abſciſſa $x\sqrt{2}$ and the parameter $a\sqrt{\frac{1}{2}}$.

SECOND SOLUTION, *by Mr.* BAZLEY.

Fig. 15, pl. 1. Suppoſe O the centre of the circle, AMB the diameter, and IR another diameter perpendicular thereto; through R draw ARK, and with the parameter AR and vertex A deſcribe a parabola to the diameter AK having its ordinates parallel to IR; the portion APpB of this parabola is the locus required.

Let CM produced, meet AR in G, and join RB. Then, ſince OR $=$ OA, AM is $=$ to MG; and by the property of the parabola and conſtruction RA $\times$ AG $=$ GP^2; but by the ſimilar triangles AMG, ARB, RA $\times$ AG $=$ BA $\times$ AM $=$ AC^2; therefore $GP^2 = AC^2$ or $GP = AC$: whence $AC - AM = GP - GM = MP$, which was to be proved.

Now from O draw Op parallel to AK to meet the curve in p; draw pT parallel to AB meeting KA produced in T; alſo draw pmg parallel to PMG meeting AO in m and AK in g. Then ſince pO is a diameter of the parabola and O the middle of AB, therefore AB is an ordinate to pO, and pT, parallel to AB, is a tangent at p, and conſequently all the ordinates except pm fall below pT; therefore pm is the greateſt ordinate; moreover, by the property of the tangent, $gA = AT$, therefore $Am = \frac{1}{2}pT = \frac{1}{2}AO$.

Finally for the area: it is well known that the area GAP is $= \frac{2}{3}GP \times AM = \frac{2}{3}AC \times AM$, therefore the area $APM = \frac{2}{3}AC \times AM - \frac{1}{2}AM \times MG = \frac{2}{3}AC \times AM - \frac{1}{2}AM^2$, which is the general area. Let BK parallel to IR meet AK in K; then it is plain that when M comes to B, the whole area APpB is $\frac{2}{3}KB \times BA - \frac{1}{2}AB \times BK = \frac{2}{3}AB^2 - \frac{1}{2}AB^2 = \frac{1}{6}AB^2$.

Mr. Dawes and J. W. *of Leeds, alſo anſwered it.*

XV. QUESTION 135, *from the* CAMB. UN. CAL.

Determine the apparent magnitude of a ſtraight rectilinear object placed at a given depth, parallel to a ſurface of water: the eye being ſituated at any point in the plane paſſing through the object perpendicular to the ſurface?

FIRST SOLUTION, *by* AMICUS.

Let AB (fig. 16, pl. 1.) repreſent the ſtraight rectilinear object placed parallel to the ſurface of the water RQ, and P the poſition of the eye in the plane ABP perpendicular to RQ, and let CLD be the curvilinear image of the object as it appears to the eye at P. See Emerson's Optics, Prop. 29. B. 3. Then the extremities A, B, of the object will appear to be elevated to the points C, D, in the perpendiculars AE, BF. Draw CP, DP, meeting EF in G and H; Join A, G; B, H; A, P; B, P; and draw GN perpendicular to FE, then CGN is equal to the angle of incidence, and AGN is the angle of refraction; therefore by the principles of optics, when the medium in which the object is immerſed is water, ſin. CGN : ſin. AGN :: 4 : 3 :: $m : n$, or, ſince the angle $CGN = GCE$ and the angle $AGN = CAG$, we have by trigonometry

$$AG : CG :: \text{ſin. } GCE : \text{ſin. } CAG :: m : n.$$

Put $PK = a$, $EK = b$, $AE = c$, and $EG = x$, then

AG²

$AG^2 = AE^2 + EG^2 = c^2 + x^2$, and by ſimilar triangles $b - x : a :: x : \frac{ax}{b-x} = CE$; hence $CG^2 = CE^2 + EG^2 = \frac{a^2x^2}{(b-x)^2} + x^2$, and $c^2 + x^2 : \frac{a^2x^2}{(b-x)^2} + x^2 :: m^2 : n^2$. By multiplying means and extremes, and reducing, we obtain $(m^2 - n^2)\,x^4 + 2(m^2 - n^2)$ $b^3 + \left\{n^2(a^2 + b^2) - m^2(c^2 + b^2)\right\} x^2 + 2m^2bc^2x - n^2c^2b^2 = 0$; from which equation the value of x may be found, when the given quantities are expreſſed in numbers, and then the point G and the diſtance GK become known. Alſo, from the ſame equation, we may determine the diſtance HF, by ſubſtituting the value of KF inſtead of b EK), and then the point K and the diſtance KH become known.

We have then only to determine the values of the angles GPH, APB; and by trigonometry

PK : KG :: 1 (rad.) : KG ÷ PK = tang. of the ∠KPG, and
PK : KH :: 1 (rad.) : KH ÷ PK = tang. of the ∠KPH;

then the ſum of the angles correſponding to theſe tangents in the tables, will be the angle GPH. And in a ſimilar way may the angle APB be calculated, and from the values of theſe angles we can determine the comparative magnitudes of the object and its image.

SECOND SOLUTION, *by Mr.* GEORGE FRANCIS, OPTICIAN, *No. 5, Old Cavendiſh Street, Cavendiſh Square, London.*

Let OB (fig. 17, pl 1.) be the object, VE and VC the diſtances of the eye and object from the ſurface of the water, and let EA, Ea, be rays incident at the points A, a, ſo as to be refracted to the given points B, O. Draw BP, Op, perpendicular to Aa, and produce EA, Ea, to meet OB produced in D, d; then OEB will be the viſual angle when the object OB is not immerſed in the fluid and DEd the viſual angle when it is immerſed. Let I : R :: $1\frac{1}{2}$: 1, then by prop. 11, of Dr. Gregory's Optics, the times of deſcribing the rays EA, AB, will be expreſſed by R·EA and I·AB, and their ſum R·AE + I·AB a minimum. Put $VE = d$, $BC = VP = a$, $VC = BP = c$, and $PA = x$; then $VA = a - x$, $EA = \sqrt{(d^2 + a^2 - 2ax + x^2)}$, $BA = \sqrt{(c^2 + x^2)}$, Eu. 47. I; and

$R \cdot EA + I \cdot AB = R\sqrt{(d^2 + a^2 - 2ax + x^2)} + I\sqrt{(c^2 + x^2)}$,

taking the fluxion, $\frac{I \times x\dot{x}}{\sqrt{(c^2 + x^2)}} + \frac{R \times (-a\dot{x} + x\dot{x})}{\sqrt{(d^2 + a^2 - 2ax + x^2)}} = 0$, and reducing, we find $(T^2 - R^2)x^4 + (2aR^2 - 2aI^2)x^3 + (I^2d^2 + I^2a^2 - R^2a^2 - R^2c^2)x^2 - 2R^2ac^2x = R^2a^2c^2$.

From this equation, the value of x, or the diſtance AP, may be found, and alſo the value of ap, by ſubſtituting pV inſtead of PV. Then, in the right angled triangles aVE, AVE, two ſides will be given in each, to determine the angles VEA, VEa, and then the ſum of theſe angles will be the viſual angle DEd.

XVI. QUESTION 136, *from the* CAMB. UN. CAL.

Let the roots of the equation $x^n - px^{n-1} + qx^{n-2} -$ &c. $= 0$, be a, b, c, &c. and thoſe of the equation $nx^{n-1} - (n-1)px^{n-2} + (n-2)qx^{n-3} -$ &c. $= 0$, be α, β, γ, &c. then, if when α, β, γ, &c. are ſucceſſively ſubſtituted in the equation $x^n - px^{n-1} + qx^{n-2} -$&c.$=0$, the reſults are P, Q, R, &c. and when a, b, c, &c. are ſubſtituted in the equation $nx^{n-1} - (n-1)px^{n-2} + (n-2)qx^{n-3} -$ &c. $= 0$, the reſults are p, q, r, &c. it will be as P × Q × R, &c. : $p \times q \times r$ &c. :: 1 : n^n. Required the demonſtration?

SOLUTION, *by* LIMENUS.

The equation of n dimenſions $= (x-a)(x-b)(x-c)\ldots$ being denoted by P, and that of $n - 1$ dimenſions $= n(x-\alpha)(x-\beta)(x-\gamma)\ldots\ldots$ by Q, the former expreſſion when the indeterminate x is of the values α, β, γ, ... which would make the latter vaniſh, is equal to $(\alpha-a)(\alpha-b)(\alpha-c)\ldots$, $(\beta-a)(\beta-b)(\beta-c)\ldots$, &c. reſpectively, and and their product is therefore equal to the product of all the exceſſes of α, β, γ, ... above a, b, c, ... taken two and two, in any order. Again, when P vaniſhes, or when x is equal to a, b, c, ..., reſpectively, the expreſſion Q is of the ſeveral values $n(a-\alpha)(a-\beta)\ldots$, $n(b-\alpha)(b-\beta)\ldots$, $n(c-\alpha)(c-\beta)\ldots$, &c. and the product of theſe is equal to n^n multiplied by the product of the exceſſes of a, b, c .... above α, β, ... taken two and two. As then theſe exceſſes are in each caſe the ſame with contrary ſigns, and their number $= n(n-1)$ is even, the above values of P multiplied together are to thoſe of Q multiplied together in the ratio of 1 to n^n.

draw rs indefinitely near and parallel to RS, alſo draw rnq parallel to MN meeting RS in n, and RO in q. Then becauſe rRq is a right angle, the triangles rnR, Rnq will be ſimilar, and rn : nR :: nR : nq, a given ratio, becauſe the poſition of the line RO is given: let this ratio be denoted by that of n to 1. Put the length of the chain BNR $= l$, NS $= x$, SR $= y$, NM $= z$, and let a denote a portion of the chain whoſe weight is equal to the tenſion at N; then rn $= \dot{x}$ and nR $= \dot{y}$; alſo put OB $= b$, and by the property of the curve, the length of the curve RN $= \sqrt{(2ax + x^2)}$, and the length of the curve BN $= \sqrt{(2az + z^2)}$; therefore BN + RN $= \sqrt{(2az + z^2)} + \sqrt{(2ax + x^2)} = l$. Again, by the property of the curve and what has been deduced $\dot{x} : \dot{y} :: \sqrt{(2ax + x^2)} : a :: n : 1$,
whence $\sqrt{(2ax + x^2)} = na$ and $x^2 + 2ax = n^2 a^2$,
completing the ſquare $x^2 + 2ax + a^2 = a^2 \times (1 + n^2)$,
taking the roots $x + a = a\sqrt{(1 + n^2)}$, and $x = a \times \left\{\sqrt{(1 + n^2)} - a\right\}$;
alſo $\dfrac{a + x + \sqrt{(2ax + x^2)}}{a} = \sqrt{(1 + n^2)} + n$.

Again from the equations $\sqrt{(2az + z^2)} + \sqrt{(2ax + x^2)} = l$, and $\sqrt{(2ax + x^2)} = na$, we get $\sqrt{(2az + z^2)} = l - na$; whence $z = \sqrt{(l^2 - 2lna + n^2a^2 + a^2)} - a$, $z - x = \sqrt{(l^2 - 2lna + n^2a^2 + a^2)}$
$- \sqrt{a^2 (1 + n^2)}$

and $\dfrac{a + z + \sqrt{(2az + z^2)}}{a} = \dfrac{\sqrt{(l^2 - 2lna + n^2a^2 + a^2)} + l - na}{a}$.

Now draw Op, Bb parallel to MN meeting RS in p and b; then we ſhall have

Op = MS = MN − NS $= z - x = \sqrt{(l^2 - 2lna + n^2a^2 + a^2)} - \sqrt{a^2(1 + n^2)}$.

Again, by known properties of the curve,

$$\text{RS} = a \times \text{h.l.}\ \frac{a + x + \sqrt{(2ax + x^2)}}{a} = a \times \text{h.l.}\ \left\{\sqrt{(1 + n^2)} + n\right\}$$

$$\text{MB} = a \times \text{h.l.}\ \frac{a + z + \sqrt{(2az + z^2)}}{a} = a \times \text{h.l.}\ \frac{\sqrt{(l^2 - 2lna + n^2a^2 + a^2)} + l - na}{a}$$

whence we ſhall have

Rp = RS + Sb − pb = RS + MB − OB

$$= a \times \text{h.l.}\ \left\{\sqrt{(1 + n^2)} + n\right\} + a \times \text{h.l.}\ \frac{\sqrt{(l^2 - 2lna + n^2a^2 + a^2)} + l - na}{a} - b.$$

Furthermore we have Rp : Op :: $n : 1$, whence Rp $= n \times$ Op, that is, by means of what has been deduced

$$a \times \text{h.l.}\ \left\{\sqrt{(1 + n^2)} + n\right\} + a \times \text{h.l.}\ \frac{\sqrt{(l^2 - 2lna + n^2a^2 + a^2)} + l - na}{a} - b$$

$$= n\sqrt{(l^2 - 2lna + n^2a^2 + a^2)} - na\sqrt{(1 + n^2)}, \text{ or}$$

$$a \times \text{h. l.} \left\{(1+n^2)+n\right\} + a \times \text{h. l.} \frac{\sqrt{(l^2-2lna+n^2a^2+a^2)}+l-na}{a}$$

$= n\sqrt{(l^2-2lna+n^2a^2+a^2)} - na\sqrt{(1+n^2)} + b$, from which expreſſion a may be found by the method of trial and error, and hence all the reſt becomes known.

XIX. QUESTION 139, *by Mr.* LOWRY.

In a given trapezium to inſcribe another, ſuch, that three of its ſides may paſs through given points and the remaining ſide be of a given length?

SOLUTION, *by the Propoſer.*

Let ABGD (fig. 20, pl. 1.) be the given trapezium, and PQRS the inſcribed one, having three of its ſides paſſing through the given points E, F, G, and the remaining ſide PS of a given length. Draw EI parallel to AB, meeting BG in I, and let the line joining the points I, F, meet GD in i; join i, C, and draw Ba parallel to Ii meeting GD in a. Let the line AE meet BG in H, and draw HF meeting GD in h and Ba produced in K; then it is evident that the points I, H, i, h, a, are given, and therefore the point K will be given, and the line KB will be given in magnitude and poſition.

Now by reaſon of the parallels,

EI : AB :: HI : HK :: IF : BK,
and EI : BP :: IQ : BQ :: IF : BL,
therefore AB : BP :: BK : BL,
and convertendo AB : AP :: BK : KL,
and permutando AB : BK :: AP : KL;

therefore, becauſe AB and BK are given lines, AP has to KL a given ratio.

Again, draw Ch, and through the point a draw akl parallel to iG meeting Ch produced in k and RS in b; then, becauſe C, i, h, a, are given points, it is evident that k will alſo be a given point, and the line ka given in magnitude and poſition.

And by the parallels, Fi : aK :: ih : ah :: iC : ak,
and Fi : aL :: iR : aR :: iC : al,
therefore aK : aL :: ak : al,
and convertendo aK : LK :: ak : kl,
and permutando aK : ak :: LK : kl,

therefore, ſince aK and ak are given lines, LK has to kl a given ratio. Let this ratio be compounded with the ratio of AP to KL, and we have

$$AB \cdot aK : BK \cdot ak :: AP : kl, \text{ a given ratio.}$$

On Ck take CV to Ck in this ratio, and draw VT parallel to kl meeting RS in T; then V will be a given point, and the line VT will be given by poſition. Then by the parallels and conſtruction CV : Ck :: VT : kl :: AP : kl; therefore VT will be equal to AP, and the problem is now reduced to the following one, *viz.*

To draw CTS to meet VT in T and AD in S, ſo that taking AP equal to VT, the line PS may be of the given magnitude. And this problem admits of diſtinct caſes, according as the line VT is parallel, or not parallel, to the ſide AD.

CASE I. When VT is parallel to AD. Fig. 21, pl. 1.

Through C draw NO parallel to AP, meeting AD in N, and VT in O, and take AI = to VO; then becauſe AP = VT, PI is = TO, and by ſimilar triangles NS : TO = PI :: NC : CO. Draw SY, IY, parallel to AI and AN, and let NY be drawn to meet AI produced in W; then SY is = PI, and YI = SP, and by the above proportion NS : SY :: (AN : AW) :: NC : CO a given ratio, becauſe NC and CO are given lines; hence we have this conſtruction:

Take AN to AW in the given ratio of NC to CO, join NW, and from I apply IY = the given line SP; draw SY parallel to AW and the point S will be determined.

CASE II. When VT is not parallel to AD. Fig. 22, pl. 1.

Let VT be produced to meet AD in L, and draw CN parallel to VN and CO parallel to AD. Take AI = to VO, and draw IY parallel to PS and SY parallel to AI. Join NY and draw NZ parallel to SY. Then becauſe AP = VT, TO is = PI = SY, and by ſimilar triangles NS : NC :: OC : TO = SY, therefore NS·SY = CN·OC = to a given rectangle, becauſe CN and OC are given lines; hence the locus of the point Y is a hyperbola whoſe centre is N and aſymptotes AN and ZN; and if this hyperbola be deſcribed, and IY = the given line PS, be applied from I to the curve at Y, then YS being drawn parallel to IA, its interſection with AD will determine the point S.

XX. QUESTION 140, *by Mr.* W. SIMPSON, *Bolton.*

$$n \times \left\{(x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4 + \text{\&c.}\ \textit{ad infinitum}\right\}$$

$$= \frac{x^n-1}{x^n} + \tfrac{1}{2}\left(\frac{x^n-1}{x^n}\right)^2 + \tfrac{1}{3}\left(\frac{x^n-1}{x^n}\right)^3 + \tfrac{1}{4}\left(\frac{x^n-1}{x^n}\right)^4 + \tfrac{1}{5}\left(\frac{x^n-1}{z^n}\right)^5 + \text{\&c.}$$

ad infinitum. Required the demonſtration?

SOLUTION.

SOLUTION, *by the Rev.* J. TOPLIS, *Arnold, Notts.*

It is well known that $\text{h. l. } (1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \&c.$ write $x - 1$ in the place of x and the expreſſion will become $\text{h. l. } (x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \&c.$ and by the nature of logarithms $\text{h. l. } (x^n) = n \times \text{h. l. } (x) = n \times \left\{ (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \&c.\right.$ *ad infinitum* $\left.\right\}$. Again it is well known that $\text{h. l. } \left(\frac{1}{1-x}\right) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \&c.$ in which expreſſion writing $\frac{x^n - 1}{x^n}$ in the place of x we ſhall have

$$\text{h. l. } (x^n) = \left(\frac{x^n-1}{x^n}\right) + \frac{1}{2}\left(\frac{x^n-1}{x^n}\right)^2 + \frac{1}{3}\left(\frac{x^n-1}{x^n}\right)^3 + \frac{1}{4}\left(\frac{x^n-1}{x^n}\right)^4 \ \&c.$$

ad infinitum. Therefore $n \times \left\{ (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \&c.\ \textit{ad infini.} \right\} = \frac{x^n-1}{x^n} + \frac{1}{2}\left(\frac{x^n-1}{x^n}\right)^2 + \frac{1}{3}\left(\frac{x^n-1}{x^n}\right)^3 + \frac{1}{4}\left(\frac{x^n-1}{x^n}\right)^4 + \&c.$ *ad infinitum.* *Q. E. D.*

XXI. QUESTION 141, *by Mr.* CUNLIFFE.

It is required to find three whole numbers, ſuch, that the ſum of every two may be a cube; and alſo that the ſum of all the three may be a cube?

SOLUTION, *by* J. W. *of Leeds.*

Let x, y, and z, denote the numbers; and by the queſtion we ſhall have

$$\begin{aligned} x + y + z &= a^3 \\ x + y &= b^3 \\ x + z &= c^3. \end{aligned}$$

The difference between twice the firſt equation and the ſum of the other two is $y + z = 2a^3 - b^3 - c^3$, which is alſo to be a cube by the queſtion.

Put $a = b + n$, and then $2a^3 - b^3 - c^3 = 2(b+n)^3 - b^3 - c$, $= b^3 + 6b^2n + 6bn^2 + 2n^3 - c^3 =$ a cube $= (b + 2n)^3$ $= b^3 + 6b^2n + 12bn^2 + 8n^3$, which gives $b = \frac{-(c^3 + 6n^3)}{6n^2}$, $a = b + n = \frac{-c^3}{6n^2}$, and $c = \frac{6cn^2}{6n^2}$; writing p for $-c$, q for n and rejecting the common denominator $6n^2$, there will be had, $a = p^3$, $b = p^3 - 6q^3$ and $c = -6pq^2$. From theſe concluſions a general ſolution to the queſtion might be obtained, but the calculation would be rather operoſe and the reſulting expreſſions complex; and beſides I have not diſcovered that a general ſolution founded upon what is here deduced has any advantage in determining numbers moderately ſmall that will anſwer the queſtion. I ſhall therefore proceed with the ſolution in a manner ſomething leſs general. For this purpoſe take $p = 3$, and $q = 1$, then $a = 27$, $b = 21$, and $c = -18$; and as each of theſe is diviſible by 3 we may take $a = 9$, $b = 7$, and $c = -6$; but c is negative, therefore in order to find poſitive numbers that will anſwer, put $a = rv + 9$, $b = sv + 7$ and $c = tv - 6$; then $2a^3 - b^3 - c^3 = 2(rv + 9)^3 - (sv + 7)^3 - (tv - 6)^3$, $= v^3(2r^3 - s^3 - t^3) + 3v^2(18r^2 - 7s^2 + 6t^2) + 3v(162r - 49s - 36t) + 1331 =$ a cube $= (gv + 11)^3 = g^3v^3 + 33g^2v^2 + 363gv + 1331$; put $3v(162r - 49s - 36t) = 363gv$ in order that two of the latter terms on each ſide of the equation may deſtroy each other; whence $g = \frac{162r - 49s - 36t}{121}$; and from the equality of the remaining terms, *viz.*

$$v^3(2r^3 - s^3 - t^3) + 3v^2(18r^2 - 7s^2 + 6t^2) = g^3v^3 + 33g^2v^2,$$

$v = \frac{33g^2 - 3(18r^2 - 7s^2 + 6t^2)}{2r^3 - s^3 - t^3 - g^3}$, where r, s, and t may be taken at pleaſure.

Take r, s, and t, each $= 1$, then $g = \frac{7}{11}$ and $v = \left(\frac{33 \times 49}{121} - 51\right) \div \left(\frac{-7}{11}\right)^3 = \left(51 - \frac{3 \times 49}{343}\right) \div \left(\frac{7}{11}\right)^3 = \frac{50094}{343}$, whence $a = rv + 9 = \frac{53181}{343}$, $b = sv + 7 = \frac{52495}{343}$, $c = tv - 6 = \frac{48036}{343}$, and rejecting the common denominator, 343, we may take $a = 53181$, $b = 52495$, and $c = 48036$.

Whence

$$
\begin{aligned}
\text{Whence}\quad a^3 &= 150407501928741\\
b^3 &= 144661785187375\\
c^3 &= 110841018670656\\
x = b^3 + c^3 - a^3 &= 105095301929290\\
y = a^3 - c^3 &= 39566483258085\\
z = a^3 - b^3 &= 5745716741366
\end{aligned}
$$

which are three numbers that will anſwer.

And thus nearly the queſtion was anſwered by Mr. Cunliffe, *the Propoſer.*

XXII. QUESTION 142, *by Mr.* CUNLIFFE.

P is a given point in the diameter AB produced of a given circle, let any line PCD be drawn cutting the circle in C and D, and let AC, BD be drawn interſecting in I: now, if in BD produced there is continually taken DR = DI, required the equation and quadrature of the curve which is the locus of R?

FIRST SOLUTION, *by Mr.* LOWRY.

Fig. 23, pl. 1. Let PF be a tangent to the circle at F and draw FG perpendicular to the diameter AB. Then AB is divided into given ſegments AE, BE, and the line FG is given by poſition and is the locus of the interſection I of the chords AC, BD (ſee the ſolutions to the prize queſtion in the 12th number of the Old Series of the Repoſitory). Hence, we have to determine the locus of the point R when the diſtance RD is conſtantly equal to the diſtance DI intercepted between the circle and the ſtraight line FG given by poſition. On AB produced take AQ=AE and BL=EB; draw OLT perpendicular to AL and join AD. Put AB $=d$, BE $=c$, BN $=x$, and RN $=y$; then BR $= \sqrt{(x^2+y^2)}$, and by ſimilar triangles,

BR : BN :: AB : BD $= \frac{dx}{\sqrt{(x^2+y^2)}}$: again, by ſimilar triangles, AB : BD :: IB : BE, and by permutation and compoſition, IB : IB+BR :: BE : BE+BN, or, AB : IB+BR :: BD : BE+BN. But ſince RD is = DI, IB + BR is = 2BI + 2DI = 2DB; therefore AB : 2DB :: DB : BE+BN $= \frac{2\text{BD}^2}{d}$, or $c+x = \frac{2dx^2}{x^2+y^2}$; this reduced gives $y = x\sqrt{\frac{2d-(c+x)}{c+x}}$ for the equation of the curve.

Now, in this equation, when x is either o. or equal to $2d - c$, that is equal to BQ, then y is o, which ſhews that the curve paſſes through the points B, Q. Alſo when x is equal to c, then $y = c\sqrt{\frac{2d-2c}{2c}} = \sqrt{(dc-c^2)} =$ EF, and the curve paſſes through the point F. In like manner, if the ordinate y be taken on the other ſide of BQ, then the curve will paſs through the points B, G, and B will be a double point in the curve. Moreover, if x be taken from B towards P, then y is $= x\sqrt{\frac{2d-(c-x)}{c-x}}$, and when x is $=$ to c (or to BL), then y is infinite, which ſhews that the line OLT is an aſymptote to the curve.

To find the area, put $z = 2d - (c+x)$, that is $=$ to QN, then y is $= \{2d-(c+z)\}\sqrt{\frac{z}{2d-z}}$, and the fluxion of the area QRN $= y\dot{z} = \dot{z}\{2d-(c+z)\}\sqrt{\frac{z}{2d-z}} = \frac{z\dot{z}(2d-z)}{\sqrt{(2dz-z^2)}} - \frac{cz\dot{z}}{\sqrt{(2dz-z^2)}} = \dot{z}\sqrt{(2dz-z^2)} - \frac{cz\dot{z}}{\sqrt{(2dz-z^2)}}$. On the diameter QL (which is equal to 2AB $= 2d$) deſcribe a circle LMQ, and let the ordinate NR meet the circle in M; then the fluent of $\dot{z}\sqrt{(2dz-z^2)}$ is the circular ſegment QMN; and the fluent of $\frac{z\dot{z}}{\sqrt{(2dz-z^2)}}$ is the arc QM $- \sqrt{(2dz-z^2)}$ $=$ the arc QM — MN; therefore the area QRN is $=$ to the ſegment QMN — (arc QM — MN) × c. And the whole area BFQGB is $=$ to the ſegment HQh — (arc HQh — Hh) × c, Hh being drawn perpendicular to AB.

Again the fluxion of the area Bnr is $= \frac{cz\dot{z}}{\sqrt{(2dz-z^2)}} - \frac{\dot{z}(2d-z)}{\sqrt{(2dz-z^2)}}$, and the fluent $=$ (arc Qm — mn) × c — ſegment Qmn — C. Now when z is $=$ QB, or $2d - c$, the area Bnr is o: but the fluent is then $=$ to the ſegment QhB — (arc Qh — hB) $=$ C; therefore the correct fluent for the area is (arc hm — ah) × c — area hBnm. And the area of the whole ſpace contained between the curve and the aſymptote indefinitely produced is $=$ to (arc HLh — Hh) × c — ſegment HLh.

SECOND SOLUTION, *by Mr.* CUNLIFFE, *the Propoſer.*

Draw the tangents PT, PT′ (fig. 24, pl. 1.) and join TT′ cutting the diameter AB in Q; then the right line TT′ is the locus of the interſection of AC and BD, this is pretty generally known. In AB produced take AV = AQ and V will be a point in the curve as appears from its generation. Draw RS, DG, perpendicular to BA and join AD; then we ſhall have QV = 2QA, and QS = 2QG, whence SV = QV − QS = 2QA − 2QG = 2 × (QA − QG) = 2AG, or AG = ½SV. Put BA = a, AQ = AV = b, VS = x, and SR = y, then AG = $\frac{x}{2}$, BG = $a - \frac{x}{2}$, BS = $a + b - x$; and per ſimilar triangles BSR, BDA;
BD² : DA² :: BS² : SR² and BD² : DA² :: BG : GA, therefore BG : GA :: BS² : SR²; whence BG × SR² = GA × BS²; that is $\left(a - \frac{x}{2}\right)y^2 = \frac{x}{2}(a+b-x)^2$ which gives $y^2 = \frac{x(a+b-x)^2}{2a - x}$, which is the equation of the curve.

The following properties are pretty obviouſly ſuggeſted by the equation.

When $x = b + a$ = VA + AB = VB, then $y = 0$, which ſhews that the curve cuts the axis again in B, the point B being what is commonly called the punctum duplex.

When $x = 2b$ = VQ, then $y^2 = 2b(a - b)^2 \div 2(a - b)$ $= b(a - b)$ = AQ × BQ = QT², which ſhews that the curve paſſes through T and T′, and this is alſo manifeſt from the generation of the curve.

When $x = 2a$, then y is infinite; therefore the curve has an aſymptote perpendicular to VB at the diſtance 2AB from V.

Again we have $y\dot{x} = \frac{(a + b)x\dot{x} - x^2\dot{x}}{\sqrt{(2ax - x^2)}}$ for the fluxion of the area. Aſſume $Ax\sqrt{(2ax-x^2)} + B\sqrt{(2ax-x^2)} + C \times \int\frac{\dot{x}}{\sqrt{(2ax-x^2)}}$ $= \int\frac{(a + b)x\dot{x} - x^2\dot{x}}{\sqrt{(2ax - x^2}}$; taking the fluxions $A\dot{x}\sqrt{(2ax - x^2)}$ $+ \frac{Ax(a\dot{x} - x\dot{x})}{\sqrt{(2ax-x^2)}} + \frac{B(a\dot{x} - x\dot{x})}{\sqrt{(2ax-x^2)}} + \frac{C\dot{x}}{\sqrt{(2ax-x^2)}} = \frac{(a+b)x\dot{x} - x^2\dot{x}}{\sqrt{(2ax - x^2)}}$:

dividing both ſides of the equation by $\frac{\dot{x}}{\sqrt{(2ax-x^2)}}$ and collecting the terms $-2Ax^2+3Aax-Bx+Ba+C = (a+b)x-x^2$; equating the coefficients of the homologous terms, $A=\frac{1}{2}$, $3Aa-B=(a+b)$, or $B=3Aa-(a+b) = \frac{a}{2}-b$, and $C=-Ba=-a\left(\frac{a}{2}-b\right)$; therefore $\int y\dot{x} = \frac{1}{2}x\sqrt{(2ax-x^2)}+\left(\frac{a}{2}-b\right)\sqrt{(2ax-x^2)}-a\left(\frac{a}{2}-b\right) \times\int\frac{\dot{x}}{\sqrt{(2ax-x^2)}} = \frac{1}{2}x\sqrt{(2ax-x^2)}+\left(\frac{a}{2}-b\right)\sqrt{(2ax-x^2)}-a\left(\frac{a}{2}-b\right)Q$, where Q is the circular arc, radius 1 and verſed ſine $=\frac{x}{2a}$.

J. W. of Leeds *alſo anſwered this Queſtion.*

XXIII. QUESTION 143, *by Mr.* BAZLEY.

Suppoſe a thread of a given length to be faſtened at one end to a given point on a perfectly ſmooth horizontal plane having a ſtraight edge; and a weight attached to the other end of the thread and brought to the edge of the horizontal plane with the thread ſtretched; then if the weight be ſuffered to deſcend by the force of gravity from this poſition; it is required to determine the time in which the thread will have acquired any other poſſible aſſigned poſition upon the plane, together with the velocity of the deſcending weight in ſuch poſition; alſo required the nature of the curve deſcribed by the deſcending weight?

FIRST SOLUTION, *by Mr.* BAZLEY.

Let AD (fig. 25, pl. 1.) be the edge of the horizontal plane, C the given point therein, CD the firſt poſition of the thread, CB any other poſition of it; and CA perpendicular to AB. Put CA $= a$, AB $= x$, and $g = 16\frac{1}{12}$ feet. Then by the reſolution of forces, $\sqrt{(a^2+x^2)} : x :: 2g : 2gx \div \sqrt{(a^2+x^2)}$ = the force whereby the point B is accelerated along the edge of the plane BA: let the velocity in B $= v$; and, by the laws of variable forces, we have, $-2gx\dot{x} \div \sqrt{(a^2+x^2)} = v\dot{v}$;

where x is written negative, becauſe v increaſes as x decreaſes; and taking the fluents, $-2g\sqrt{(a^2+x^2)}=\frac{1}{2}v^2$, or, $-4g\sqrt{(a^2+x^2)}=v^2$: but when $x=\text{AD}=r$, then $v=0$; whence, by correction, $4g(\sqrt{(a^2+r^2)}-\sqrt{(a^2+x^2)})=v^2$, or putting $\sqrt{(a^2+r^2)}=\text{CD}=d$, we obtain $v=2g^{\frac{1}{2}}\sqrt{\left\{d-\sqrt{(a^2+x^2)}\right\}}$. Again, for the time, we ſhall have,

$$\dot{t}=\frac{-\dot{x}}{2g^{\frac{1}{2}}\sqrt{\left\{d-\sqrt{(a^2+x^2)}\right\}}},$$

where $\dot{x}$ is negative for the ſame reaſon as before. Now aſſume

$$\frac{-\dot{x}}{2g^{\frac{1}{2}}\sqrt{\left\{d-\sqrt{(a^2+x^2)}\right\}}}=\dot{x}\times(\text{A}x+\text{B}x^2+\text{C}x^4+\text{D}x^6+\&\text{c.}),$$

and reduce the whole expreſſion by involution, &c. to ſimple terms, and we ſhall find

$$\text{A}=-\frac{1}{c^{\frac{1}{2}}},\ \text{B}=-\frac{1}{4ac^{\frac{3}{2}}},\ \text{C}=-\frac{3a-2c}{32a^3c^{\frac{5}{2}}},\ \text{D}=-\frac{5a^2-6ca+4c^2}{128a^5c^{\frac{7}{2}}},\ \&\text{c.}$$

whence, taking the fluents, $t=\text{A}x+\frac{\text{B}x^3}{3}+\frac{\text{C}x^5}{5}+\frac{\text{D}x^7}{7}+$ &c. wherein taking the values of A, B, C, D, &c. as found above, and making the ſeries (by correction) to vaniſh when $x=0$, we ſhall obtain the correct value of t.

And the velocity of the deſcending weight at any point will be readily obtained from what has been deduced; for the velocities in any direction are directly as the fluxions in the reſpective directions; therefore, fluxion of $\text{BA}=-\dot{x}$: fluxion of $\text{BW}=-x\dot{x}\div\sqrt{(a^2+x^2)}::\sqrt{(a^2+x^2)}:x::$

$$v=2g^{\frac{1}{2}}\sqrt{\left\{d-\sqrt{(a^2+x^2)}\right\}}:2g^{\frac{1}{2}}x\sqrt{\frac{d-\sqrt{(a^2+x^2)}}{a^2+x^2}}.$$

which expreſſes the velocity of the deſcending weight at W.

As to the curve deſcribed by W, it is the equilateral hyperbola. For ſince BW is perpendicular to AB, it will ſtill be ſo if we conceive the plane of the curve to be raiſed up, till it coincides with the given horizontal plane; then let CA be produced till AG = CD, draw GF parallel to AD, and produce BW to F. Now by the deſcription of the curve, CB + BW = CD = BF by conſtruction; therefore CB = FW, and $\text{CB}^2=\text{FW}^2=\text{AC}^2+\text{AB}^2$: let the curve meet CG in V; then CA = VG, and $\text{FW}^2=\text{GV}^2+\text{GF}^2$, which is the property of the equilateral hyperbola whoſe centre is G and vertex V.

SECOND SOLUTION, *by Mr.* LOWRY.

Let AD (fig. 25, pl. 1.) be the ſtraight edge of the horizontal plane, C the point where the end of the thread is faſtened, CD the poſition of the thread at the commencement of the motion, or when the weight is at D, the edge of the plane. Draw AC perpendicular to AD and put AC $= a$, AD $= b$, CD $= d$, AB $= x$, and $g = 32\frac{1}{6}$ feet the meaſure of the accelerative force of gravity. Then the force accelerating the thread towards A at the point B, being to the force in the direction CB, as AB to BC, we have, CB : AB :: g : $gx \div \sqrt{(a^2 + a^2)} =$ the accelerative force in the direction AB; therefore by the known principles of dynamics

$$\frac{\ddot{x}}{\dot{t}^2} + g \cdot \frac{x}{\sqrt{(a^2 + x^2)}} = 0$$

multiply by $2\dot{x}$ and take the fluents, and we have

$$\frac{\dot{x}^2}{\dot{t}^2} + 2g\sqrt{(a^2 + x^2)} = c.$$

To determine c, the velocity or $\frac{\dot{x}}{\dot{t}}$ is $= 0$, when $x = b$ therefore c is $= 2g\sqrt{(a^2 + b^2)} = 2gd$, and

$$\frac{\dot{x}^2}{\dot{t}^2} = 2gd - 2g\sqrt{(a^2 + x^2)}.$$

$$\text{or, } \frac{\dot{x}}{\dot{t}} = (2g)^{\frac{1}{2}}\sqrt{\left\{d - \sqrt{(a^2 + x^2)}\right\}}$$

$$\text{and } \dot{t} = \frac{\dot{x}}{(2g)^{\frac{1}{2}}\sqrt{\left\{d - \sqrt{(a^2 + x^2)}\right\}}}.$$

The fluent of this expreſſion, may, by a very ſimple transformation, be exhibited by means of elliptic arcs, or it may be found by a ſeries as is done at page 200 of Dr. Hutton's Select Exerciſes and conic ſections.

Now $\frac{\dot{x}}{\dot{t}} = (2g)^{\frac{1}{2}}\sqrt{\left\{d - \sqrt{(a^2 + x^2)}\right\}}$ is the velocity in the direction DB, and the velocity in the direction BW is to the velocity in the direction DB as the fluxion of BW to the fluxion of DB; that is, as $\dot{x}$ to $x\dot{x} \div \sqrt{(a^2 + x^2)}$, therefore the velocity with which the weight deſcends is

$$= \sqrt{\frac{2gx^2}{a^2 + x^2}} \times \sqrt{\left\{d - \sqrt{(a^2 + x^2)}\right\}}.$$

To find the nature of the curve deſcribed by the deſcending weight. put BW $= y$, then CB $= \surd(a^2 + x^2)$, and ſince CB + BW = CD $= d$, we have BW $= d - \surd(a^2 - x^2)$, and by reduction $x^2 = b^2 - 2dy + y^2$, an equation to the equilateral hyperbola.

XXIV. QUESTION 144, *by Mr.* CUNLIFFE.

Having given a triangle, it is required to determine a circle and concentrical ellipse, together with the position of a right line paſsing through the ſaid centre, such, that one of the angles of the triangle being always placed upon the right line, the other two angles ſhall be one in the circumference of the circle, and the other in the periphery of the ellipſe ?

FIRST SOLUTION, *by Mr.* LOWRY.

Let RFB (fig. 26, pl. 2.) be the given triangle, and take RH and CR each equal to the side FR. With the centre C, and radius CR, describe a circle RI; with the semiaxes CB, HB, describe an ellipse DBE, and through the given points C, F, draw the straight line CP; then RI is the circle, and DBE the ellipse which were to be found, and CP is the ſtraight line given by position.

Let MKG be a triangle equal to RFB, having the angle K (F) placed at any point on the line CP, and the angle M (B) placed in the periphery of the ellipſe; then we have only to prove that the point G (R) is in the circumference of the circle RI. Produce MG till MT = CB, and join TK; then the triangles MTK, BCF will be equal in all reſpects, and it is proved, in the ſecond ſolution to the 88th. queſtion in Vol. I, that the point T is in the axis CD, and that TO is = to CH = 2CR. But MG is = BR and BC = MT, therefore TG is = CR; and, conſequently, GO is alſo = to CR, and TO is biſected in G. Therefore, becauſe TCO is a right angle, G is the centre, and GC the radius of the circle which paſſes through the points T, C, O, that is, GC is = to $\frac{1}{2}$TO = CR, and, conſequently, the point G is in the circumference RI deſcribed with the radius CR.

SECOND SOLUTION, *by Mr.* CUNLIFFE, *the Propoſer.*

Let CA, CB (fig. 27, pl. 2.) be two right lines given by position, and EF a right line of a given length having its extremes E and F always in the lines CA, CB; then the

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locus of any point G in EF will be an ellipſe whoſe centre is C; this is very well known. Draw ED, FD, reſpectively perpendicular to CA, CB meeting each other in D, and draw CD, which biſect in O, and join EO, FO, and GO.

Then CED and CFD being right angles, a circle whoſe diameter is CD will paſs through E and F; and CD is given, becauſe EF and the angle ECF are given; hence $CO = OE = OF = \frac{1}{2}CD$ is given, therefore the locus of O is a circle whoſe centre is C. Wherefore the triangle EGO will have the angle E upon the right line CA, the angle O in the circumference of the circle whoſe centre is C and radius $CO = OE$: and the angle G in the periphery of an ellipſe whoſe centre is C, and principal ſemi-axes $OC + OG$ and $OE - OG$; and the angle formed by the ſemi-tranſverſe axis and the line CA equal to half the angle EOG; this appears from my ſolution to queſtion 113.

And the determination of the circle, concentrical ellipſe, and poſition of the right line paſſing through the common centre for a given triangle will eaſily follow.

For let EOG (fig. 28, pl. 2.) be a given triangle; E the angle to be placed on the right line; O the angle to be placed in the circumference of the circle; and G the angle to fall in the periphery of the ellipſe.

Produce GO till $OC = OE$ and join CE; with the centre C and radius CO deſcribe a circle: alſo with the centre C, ſemi-tranſverſe GC and ſemi-conjugate equal to $OE - OG = OC - OG$ deſcribe an ellipſe; then the angle E being always placed on the right line CE and the angle O in the circumference of the circle whoſe centre is C and radius $CO = OE$ (the planes of the triangle, circle and ellipſe coinciding), the angle G will fall in the periphery of the ellipſe.

XXV. QUESTION 145, *by Mr.* I. I.

If from a point without an ellipſe or an hyperbola, two tangents be drawn to the curve, and likewiſe two ſtraight lines be drawn to the foci, the angle contained by one tangent and the line drawn to one focus will be equal to the angle contained by the other tangent and the line drawn to the other focus?

SOLUTION, *by Mr.* LOWRY.

Let PG, pG, be two tangents drawn from the point G to an ellipſe or hyperbola; F, f, the foci, and let the lines GF, Gf, be drawn; then in the ellipſe (fig. 29, pl. 2.) the angle

PGf = pGF, and the angle PGF = pGf. And in the hyperbola (fig. 30, pl. 2.), fG being produced to L, the angle PGL = pGF, and the angle PGF = pGL.

Draw FQ, FH perpendicular to the tangents, meeting fP, fp (or theſe lines produced) in Q and H, and join QG, HG. Then, becauſe the tangent PG biſects the angle QPF, it is evident that the right angled triangles PIF, PIQ, are equal in all reſpects; therefore PF = PQ; and the triangles GPQ, QPF, having two ſides and the included angles reſpectively equal, they will be equal in every reſpect; therefore GQ = GF, and the angle PGQ = PGF. In the ſame way it is ſhewn that GH = GF, and the angle pGH = pGF; therefore GQ is = to GH, and fQ is = to fH (each being equal to the tranſverſe axis AB), therefore the triangles fQG, fHG, are equal in every reſpect, and the angle fGQ = FGH.

Now, in the ellipſe, the angle fGQ is = PGf + PGF, and the angle fGH = pGf + pGF; therefore PGf + PGF = pGf + pGF: take the angle fGF from each, and we have 2PGf = 2pGF, or PGf = PGF. Let the angle fGF be added to each of theſe and then the angle PGF = pGF.

Again, in the hyperbola, becauſe the angle fGQ = fGH, the angle QGL is = to HGL: but the angle QGL is = PGF + PGL, and the angle HGL is = pGF + pGL; therefore PGF + PGL = pGF + pGL; take the angle FGL from each and we have 2PGL = 2pGF, or PGL = pGF. Let the angle FGL be added to each of theſe and we have the angle PGF = pGL.

A property analogous to that which we have juſt demonſtrated for the ellipſe and hyperbola, belongs alſo to the parabola; for if tangents GP, Gp be drawn to a parabola whoſe focus is F, and Gf be a diameter, then the angle PGf will be equal to the angle pGF. (fig. 31, pl. 2.)

Let the lines be drawn as in the other caſes, and join Pp, QH meeting Gf in f and S: then it is proved in the ſame way as before, that the angle FGI = QGI, the angle FGK = KGH, and QG = GH. But Pf is = pf, (becauſe Pp is an ordinate to Gf), therefore QS is = SH; and the triangle QGH being iſoſceles, GS is perpendicular to QH, therefore the angle fGQ is = to the angle fGH. The remaining part of the demonſtration will now be the ſame as for the ellipſe.

Mr. Bazley *alſo favoured us with an anſwer to this queſtion.*

XXVI. QUESTION 146, *by Mr.* LOWRY.

Let there be any number of ſtraight lines AP, BP, CP, DP, &c. given by poſition, and let Z be a given point without them; it is required to draw a circle through the

points

points P and Z, to interſect the given lines in A, B, C, D, &c, ſo that the ſum of all the intercepted parts AP, BP, CP, DP, &c. may be equal to a given line?

First Solution, *by Mr.* Lowry, *the Propoſer.*

Draw PQ (fig. 32, pl. 2.) to biſect the angle APB and meet the circle in Q, then PQ is given by poſition. Aſſume any point a, as given, in PA, and draw ab parallel to PB meeting PQ in b, then b is a given point. Make bc parallel to PC, and equal to Pa; then c is a given point, and the line Pc drawn to meet the circle in R, is given by poſition. Make cd parallel to PD, and equal to Pb; then d is a given point, and the line Pd drawn to meet the circle in S, is given by poſition. Proceed in this manner till lines have been drawn parallel to all the lines given by poſition. Join AQ, BQ, CQ, RQ, SQ, RD, &c. then, becauſe ab is parallel to PB, and the angle APQ (ABQ) $=$ QPB (QAB), the triangles aPb, QBA, are equiangular, therefore $AB : AQ :: Pb : Pa$. Again, becauſe bc is parallel to PC, the angle bcP is $=$ to the angle RPC or RQC, and the angle bPc or QPR is $=$ to QCR, therefore the triangles RCQ, Pbc, are equiangular;

and $RC : QR :: Pb : bc = Pa$; and $QC : QR :: Pc : Pb$.

It may be proved in the ſame way that the triangles RDC, Pdc, are equiangular, and, therefore, $SD : RS :: Pc : dc = Bb$, and $RD : RS :: Pd : cd$.

Now, Playfair's Euclid, Prop. E. B. VI.

$PA + PB : PQ :: AB : AQ :: Pb : aP :: RC : QR$, or $(PA+PB)QR = PQ \cdot RC =$ (Prop. D. ibid.) $PR \cdot QC - PC \cdot QR$; therefore $(PA + PB + PC)\, QR = PR \cdot QR$,

or, $PA + PB + PC : PR :: QC : QR$; therefore $(PA+PB+PC)RS = PR \cdot SD =$ (prop. D. ibid.) $PS \cdot RD - PD \cdot RS$, and $(PA + PB + PC + PD)\, RS = PS \cdot RD$,

or, $PA + PB + PC + PD : PS :: RD : RS :: Pd : cd$.

We may proceed in the ſame manner for any number of lines whatever, and the method of conſtruction is now very obvious. Thus, if there be two lines AP, BP, and S be the given ſum, let PQ be drawn to biſect the angle APB, and take QP to S in the given ratio of $aP : Pb$; then a circle deſcribed through the points, P, Q, Z, will be that required. If there be three lines AP, BP, CP, let PR be drawn as in the analyſis, and take PR to S in the given ratio of $Pb : Pc$; deſcribe a circle through the points P, R, Z, and the problem will be done. If there be four lines

AP, BP, CP, DP, let PS be drawn, and make PS to S in the given ratio of cd : Pd; then the circle paſſing through the points P, S, Z, will be the one required.

The circle, inſtead of paſſing through a given point Z, may either touch a ſtraight line or a circle given by poſition, or have its centre in a line of any kind given by poſition.

The analyſis of the problem will be nearly the ſame when the ſum of the rectangles contained by the lines PA, PB, PC, PD, &c. and given lines a, b, c, d, &c. is equal to a given ſpace.

The following poriſm may alſo be immediately deduced from what has been done above.

Let there be any number of ſtraight lines PA, PB, PC, PD, &c. given by poſition and interſecting in the ſame point P; a point may be found, ſuch, that if any circle whatever be deſcribed through the point P and the point found, and meeting the ſtraight lines given by poſition in the points A, B, C, D, &c.; the ſum of the intercepted parts AP, BP, CP, DP, &c. ſhall be equal to a given line: or the ſum of the rectangles $a \cdot AP$, $b \cdot BP$, $c \cdot CP$, $d \cdot DP$, &c. ſhall be equal to a given ſpace.

SECOND SOLUTION, *by Mr. J. W. of Leeds.*

From the given point Z (fig. 33, pl. 2.) draw $Z\alpha$, $Z\beta$, $Z\gamma$, $Z\delta$, &c. perpendicular to PA, PB, PC, PD, &c. and join PZ; then having biſected PZ in X with the indefinite perpendicular Xx, and taken L equal to the line, let OX be taken in Xx ſo that $Z\alpha + Z\beta + Z\gamma + Z\delta$, &c. : $L - P\alpha - P\beta - P\gamma - P\delta$, &c. :: PX : OX; join OP, OZ, and a circle deſcribed about the centre O with the radius OP or OZ will be that which was required to be deſcribed.

Let the circles meet the ſtraight lines given by poſition in
A, B, C, D, &c. and join ZA, ZB, ZC, ZD, &c. It is evi-
dent that the triangles $Z\alpha A$, $Z\beta B$, $Z\gamma C$, &c. are each of them
ſimilar to the triangle PXO: hence
$PX : OX :: Z\alpha : A\alpha :: Z\beta : B\beta :: Z\gamma : C\gamma :: Z\delta : D\delta$, &c. wherefore
$Z\alpha + Z\beta + Z\gamma + Z\delta$ &c. : $A\alpha + B\beta + C\gamma + D\delta$ &c. :: PX : OX;
but by the conſtruction
$Z\alpha + Z\beta + Z\gamma + Z\delta$ &c. : $L - P\alpha - P\beta - P\gamma - P\delta$ &c. PX : OX;
therefore $A\alpha + B\beta + C\gamma + D\delta$ &c. $= L - P\alpha - P\beta - P\gamma - P\delta$ &c.
wherefore $A\alpha + B\beta + C\gamma + D\delta$ &c. $+ P\alpha + P\beta + P\gamma + P\delta$ &c. $= L$;
that is, the ſum the lines PA, PB, PC, PD, &c, is equal to L
the given line by conſtruction.

XXVII. QUESTION 147, *by Mr.* WALLACE, *R.M. College.*

If x denote an arch of the meridian, and y denote the meridional parts correſponding to the latitude x, then

$$y = x + \frac{x^3}{6} + \frac{x^5}{24} + \frac{61x^7}{5040} + \frac{277x^9}{725678} + \&c.$$

and by reverting the ſeries

$$x = y - \frac{y^3}{6} + \frac{y^5}{24} - \frac{61y^7}{5040} + \frac{277y^9}{725678} - \&c.$$

where it is remarkable that the numeral coefficients of the powers of x are the ſame as thoſe of the like powers of y, and the ſigns of the terms of the one ſeries are all $+$, but thoſe in the other ſeries $+$ and $-$ alternately. Shew the reaſon of theſe peculiarities?

SOLUTION, *by Mr.* WALLACE, *the Propoſer.*

It is well known that the relation between x and y is expreſſed by the equation

$$y = \int \frac{\dot{x}}{\text{coſ. } x} = \tfrac{1}{2} \text{ hyp. log. } \frac{1 + \text{ſin. } x}{1 - \text{ſin. } x},$$

therefore, putting e for the number whoſe hyp. log. $= 1$,

$$e^{2y} = \frac{1 + \text{ſin. } x}{1 - \text{ſin. } x}$$

and hence $\text{ſin. } x = \dfrac{e^{2y} - 1}{e^{2y} + 1}$,

and $\text{coſ. } x = \dfrac{2e^{y}}{e^{2y} + 1} = \dfrac{1}{\frac{1}{2}(e^{y} + e^{-y})}$,

and as from the fluxional equation $\dot{y} = \dfrac{\dot{x}}{\text{coſ. } x}$ we have

$\text{coſ. } x = \dfrac{\dot{x}}{\dot{y}}$, therefore

$$\frac{\dot{x}}{\dot{y}} = \frac{1}{\frac{1}{2}(e^{y} + e^{-y})}, \text{ and } \dot{x} = \frac{\dot{y}}{\frac{1}{2}(e^{y} + e^{-y})};$$

But $\text{coſ.}\, x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} \quad \frac{x^6}{1.2.3.4.5.6} + \&c.$

and $e^y = 1 + y + \frac{y^2}{1.2} + \frac{y^3}{1.2.3} + \frac{y^4}{1.2.3.4} + \frac{y^5}{1.2.3.4.5} + \&c.$

and $e - y = 1 - y + \frac{y^2}{1.2} - \frac{y^3}{1.2.3} + \frac{y^4}{1.2.3.4} - \frac{y^5}{1.2.3.4.5} + \&c.$

and therefore $\frac{1}{2}(e^y + e^{-y}) = 1 + \frac{y^2}{1.2} + \frac{y^4}{1.2.3.4} + \frac{y^6}{1.2.3.4.5.6} + \&c.$

Put $\frac{1}{1.2} = a$, $\frac{1}{1.2.3.4} = b$, $\frac{1}{1.2.3.4.5.6} = c$, &c.

then ſubſtituting theſe values of coſ. x and $\frac{1}{2}(e^y + e^{-y})$ in the fluxional equations

$$\dot{y} = \frac{\dot{x}}{\text{coſ.}\, x}, \quad \dot{x} = \frac{\dot{y}}{\frac{1}{2}(e^y + e^{-y})},$$

they become

$$\dot{y} = \frac{\dot{x}}{1 - ax^2 + bx^4 - cx^6 + \&c.},$$

$$\dot{x} = \frac{\dot{y}}{1 + ay^2 + by^4 + cy^6 + \&c.}$$

Now if v denote any quantity, by known methods we find

$$\frac{1}{1 - av + bv^2 - cv^3 + \&c.} = 1 + Av + Bv^2 + Cv^3 + \&c.$$

where A, B, C, &c. denote certain conſtant numbers that are independent of the value of v, therefore the ſame equation muſt hold true whatever quantity be ſubſtituted inſtead of v. If we ſubſtitute x^2 then we get

$$\frac{1}{1 - ax^2 + bx^4 - cx^6 + \&c.} = 1 + Ax^2 + Bx^4 + Cx^6 + \&c.$$

and if we put $-y^2$ inſtead of v we get

$$\frac{1}{1 + ay^2 + by^4 + cy^6 + \&c.} = 1 - Ay^2 + By^4 - Cy^6 + \&c.$$

therefore the two equations

$$\dot{y} = \frac{\dot{x}}{1 - ax^2 + bx^4 - cx^6 + \&c.}$$

$$\dot{x} = \frac{\dot{y}}{1 + ay^2 + by^4 + cy^6 + \&c.}$$

are

all reſpects; therefore GK is equal to OI, a given line; and G being a given point, the locus of the point K will be a circle mKn, given by poſition.

Now, when the circles mKn, HKR, interſect each other, there will be two points K, where the angle IpK will be equal to the given one: but when they only touch each other, there can be one point K only, and the angle IpK will then be the leaſt poſſible. For if it was leſs, the circle mKn could not meet the circle HKR, and the problem would be impoſſible.

When the circles touch at K, the line KG paſſes through the centre C, and CG being equal to OI or OA, and CK equal to CH, CG will be equal to the difference of the radii CH, OA (equal to CO + AH), and Gp being equal to pO we have the following ſimple conſtruction:

With the diſtances pG (pO), and GC (CO + AH), and centres p, C, deſcribe two arcs to interſect at G, produce CG to meet the exterior circle in K, and pK is the radius of the circle required.

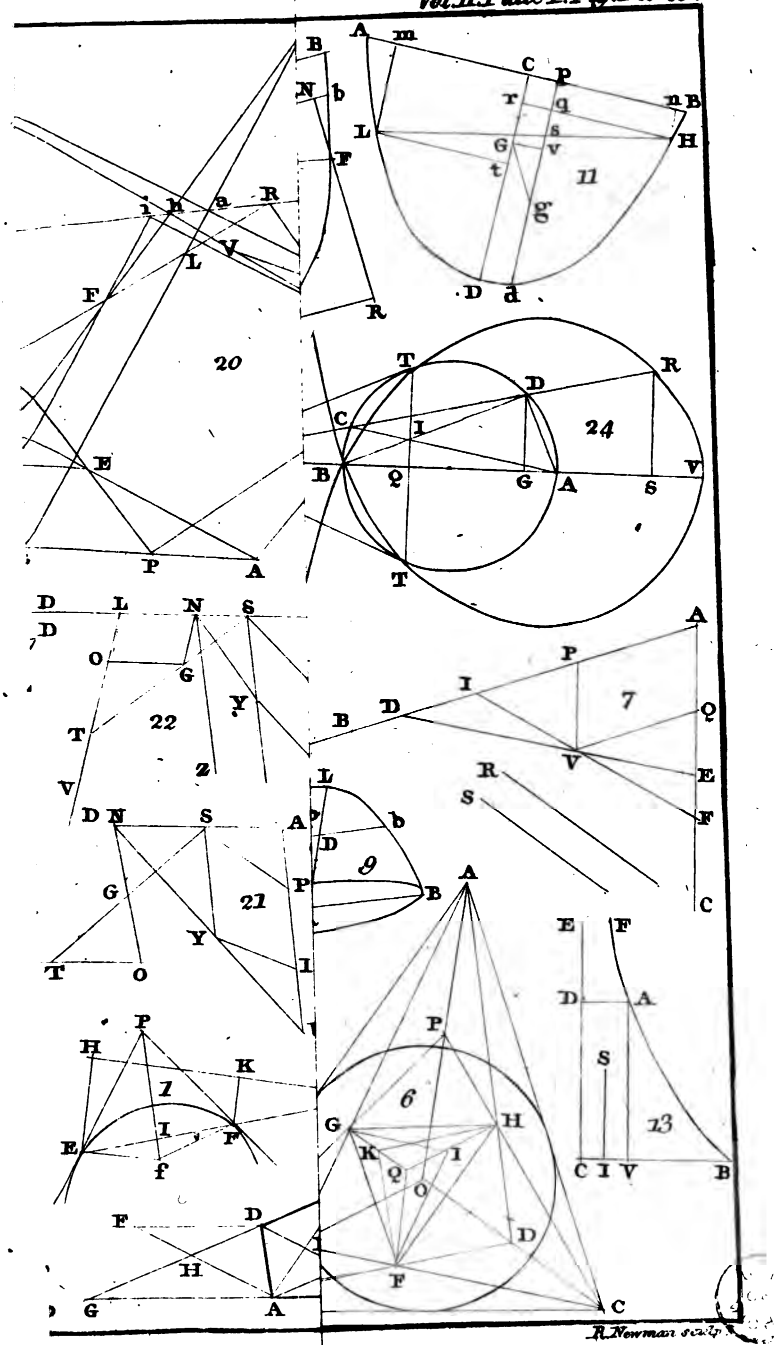
The preceding analyſis and conſtruction may be applied to the ſolution of an aſtronomical problem which has often engaged the attention of mathematicians; *viz.* To find on what day of the year the duration of twilight is the ſhorteſt in a given latitude.

For, conceive a plane, perpendicular to the horizon H′R′ (fig. 36, pl. 2.), to paſs through the poles P, P′, and meet the plane of the equator in E. Q., and the plane of a ſmall circle 18° below the horizon, and parallel to it, in a, b. Draw PH′, Pa, PP′, Pb, PR′, meeting the plane of the equator in H, A, p, B, R; then theſe points will evidently be the projections of the points H′, a, P′, b, R′ on that plane reſpectively. Alſo, the angle H′PP′ being meaſured by half the arc H′P′, the tangent Hp of the angle H′PP′ (to the radius pP) will be the tangent of half the arc H′P′ or PR′, the latitude of the place. For a like reaſon pR will be the tangent of half the arc P′R′ = tang. $\frac{1}{2}$ (90° — $\frac{1}{2}$ lat:); pA the tangent of half the arc aP′ = tang. $\frac{1}{2}$ (lat. — 18°) = tang. ($\frac{1}{2}$ lat. — 9°); and pB = the tangent of half the arc P′*b* = tang.

$$\left\{90^\circ - \tfrac{1}{2}(\text{lat.} + 18^\circ)\right\} = \text{tang. } (81^\circ - \tfrac{1}{2}\text{ lat.}).$$

Biſect HR in C, and join PC; then becauſe HPR is a right angle, PC is = to CH or CR, and the angle HCP = to twice the angle PRC = to twice the angle HPp = to the angle P′pH′ = to the angle NpQ; therefore pC, the tangent of the angle P′PC, is the tangent of the arc P′N, the co-latitude; and PC is the ſecant of the co-latitude.

Now, let the diſtances from p (in fig. 36,) be tranferred to 5, by taking pQ = ~~p P for the~~ radius of the primitive circle, Vol. II. Plate I. Fig. 1 to 25.



R. Newman sculp.

circle, pC = the co-tangent, and CH = the co-ſecant of the given latitude: with the centre C and diſtance CH, deſcribe a circle HKB, which will repreſent the horizon, when projected on the plane of the equator. Again, take Ap = the tang. ($\frac{1}{2}$ lat — 9°), and Bp = (81° — $\frac{1}{2}$ lat.) and on the diameter AB deſcribe a circle AIB which will be the projection of the ſmall circle at 18° from the horizon, where the twilight begins or ends. Biſect AB at O, and draw CK as in the foregoing conſtruction; then the diſtance pK. will be the tangent of half the co-declination. For the arc KI, deſcribed about the pole p, will be the projection of the parallel of declination, pK and pI the projections of the azimuth circles at the beginning and end of twilight, and the angle KpI will be equal to the angle included between thoſe circles, and which angle is proportional to the duration of twilight, and therefore, when the angle KpI is the leaſt poſſible, the duration of twilight will be the ſhorteſt.

When the latitude is 18°, the points A and p coincide, and when the latitude is leſs than 18°, the point A will fall between p and B, but the conſtruction will ſtill be the ſame as the above. When Hp is = to pB, the points K, I, will coincide with H, B, and tang. $\frac{1}{2}$ lat. = tang. (81° — $\frac{1}{2}$ lat.) or the latitude = 81°, and the twilight will continue twelve hours.

The calculation may be eaſily made out from the conſtruction, for in the triangle pGC, the three ſides are given to find the angle at G, and then in the triangle pGK, two ſides and the included angle are given to find the ſide pK.

XXIX. QUESTION 149, *by* MECHANICUS.

To place a ſtraight rod, or beam, loaded with any given weights, ſo that it may be in equilibrio, reſting upon an upright prop, and one end touching a ſmooth vertical wall given by poſition?

FIRST SOLUTION, *by Mr.* LOWRY.

Let AB (fig. 37, pl. 2.) repreſent the vertical plane or wall, ID the upright prop, KB the poſition of the rod when it reſts in equilibrio, and G the centre of gravity of the rod and weights. Draw GE parallel, and BE perpendicular to AB, and join EI. Then, the forces that act on the rod are theſe three; the force of gravity, equal to the whole weight, acting at G in the vertical direction GE; the force ariſing from the preſſure of the prop againſt the rod at I, acting in a direction perpendicular to

KB;

KB; and, thirdly, the force ariſing from the preſſure of the wall againſt the end of the rod at B, which acts in the direction BE. And that the rod may reſt in equilibrio, it is neceſſary that the directions of theſe forces tend to the ſame point; therefore EI will be the direction of the force acting at I, or EI will be perpendicular to the rod KB. Becauſe ID is parallel to GE, the forces at G, I, and B, are proportional to the lines ID, IE, and ED; or, if IE repreſent the abſolute force at I, and be reſolved into the two forces ED, ID; then the force ED is that part of the force IE which is equal and oppoſite to the preſſure at B, and prevents the rod from moving horizontally; and the force ID is that part of the force IE which is equal and oppoſite to the force of gravity at G, and hinders the rod from deſcending in a vertical direction.

To find the poſition of the rod, let IC be drawn perpendicular to AB, and produce EI to meet the plane at A; then by ſimilar triangles GB : BI :: EB : BD = IC :: AB : AC :: AB·BC : AC BC: but AB·BC $=$ BI^2, and AC·BC $=$ IC^2; therefore GB : BI :: BI^2 : IC^2, or BI^3 $=$ GB·IC^2. Put $l =$ to the length of the rod; $b =$ to IC, the diſtance of the prop from the wall; $m =$ the weight of the rod; w, w', &c. the weights placed at the diſtances d, d', &c. from the end B of the rod; $a =$ GB, the diſtance of the common centre of gravity of the rod and weights from the ſame end B; and $x =$ IB. Then, by the ordinary rule for finding the centre of gravity of weights placed in a ſtraight line, we have

$$\text{GB} = a = \frac{\frac{1}{2}lm + dw + d'w' + \text{\&c.}}{m + w + w' + \text{\&c.}} = \frac{\frac{1}{2}lm + \text{W}'}{m + \text{W}},$$

where $\text{W}' = dw + d'w' + \text{\&c.}$ and $\text{W} = w + w' + \text{\&c.}$

$$\text{Therefore } x^3 = b^2 \times \frac{\frac{1}{2}lm + dw + d'w' + \text{\&c.}}{m + w + w' + \text{\&c.}} = b^2 \times \frac{\frac{1}{2}lm + \text{W}'}{m + \text{W}}.$$

If $m = 0$, or the rod has no weight, and a ſingle weight w only be placed at the extremity of the rod; then $x^3 = b^2 \times \frac{dw}{w} = b^2 \times l$, which ſhews that the poſition of the rod will be the ſame, whatever the weight w may be, which is affixed to the end of it.

To compare the preſſure at I and B with the ſum of the weights $m + w + w' +$ &c. or the force at G, let P, P′ be put to denote theſe preſſures reſpectively; then the triangles IBC, IED, being ſimilar, IC : IB :: ID : IE :: $m + \text{W} : \text{P} = \frac{\text{IB}}{\text{IC}} \times$

$(m + \mathrm{W}) = \sqrt[3]{\frac{a}{b}} \times (m + \mathrm{W})$; and IC : BC :: ID : IE ::

$m + \mathrm{W} : \mathrm{P}' = \frac{\mathrm{BC}}{\mathrm{IC}} \times (m + \mathrm{W}) = \sqrt{(\sqrt[3]{a^2 b} - b)} \times \frac{m + \mathrm{W}}{\sqrt{b}}$.

This ſolution is on the ſuppoſition that the plane or wall is perfectly ſmooth, or void of friction: but if the friction be taken into account, and be ſuppoſed proportional to the preſſure, the ſolution may then be as follows. Draw BF (fig. 38, pl. 2.) perpendicular to BA, and take BL to BF, in the given proportion of the friction to the preſſure (n : 1); complete the parallelogram BFRL, and draw BR meeting CE, a parallel to AB at E; then ſince the friction oppoſes the aſcent of the rod at B, it may be conſidered as a force acting in the direction BL; therefore BR will be the direction of a force, equivalent to the two forces of friction and preſſure acting in the directions BL, BF. Let EN be drawn parallel to BF meeting ID produced in N; then, the rod is kept in equilibrio, in the ſame manner as before, by three forces acting in the directions GE, IE, BE; and proportional to the lines ID, IE, ED; or, if IE repreſent the abſolute force at I, and be reſolved into two forces EN, DN: then EN will be equal and oppoſite to the force of preſſure at B, and will hinder the rod from moving in a horizontal direction; alſo DN will be equal to the force ariſing from the friction of the plane and will aſſiſt the force of gravity DI in oppoſing the vertical force IN, ſo as to prevent the rod from aſcending or deſcending in a vertical direction,

To find the poſition of KB, draw IP parallel to EB; then by ſimilar triangles, &c. as in the former caſe, we have $\mathrm{BI}^3 = \mathrm{GB}\cdot\mathrm{BC}\cdot\mathrm{AP}$. But $\mathrm{AP} = \mathrm{AC} - \mathrm{PC}$, and $\mathrm{BC}\cdot\mathrm{AP} = \mathrm{BC}\,(\mathrm{AC} - \mathrm{PC}) = \mathrm{IC}^2 - \mathrm{BC}\cdot\mathrm{PC}$, therefore $\mathrm{BI}^3 = \mathrm{GB}\,(\mathrm{IC}^2 - \mathrm{BC}\cdot\mathrm{PC})$.

Becauſe the triangles ICP, RLB, are ſimilar, PC : IC :: BL : LR or BF :: n : 1, therefore $\mathrm{PC} = n\cdot\mathrm{IC} = nb$, and $bc = \sqrt{(x^2 - b^2)}$.

Therefore $x^3 = a \times \left\{b^2 - nb\sqrt{(x^2 - b^2)}\right\}^*$,

or $x^6 - 2ab^2x^3 - n^2a^2b^2x^2 + (n^2 + 1)\,a^2b^4 = 0$.

Subſtitute $\frac{\frac{1}{2}lm + dw + d'w' + \&c.}{m + w + w' + \&c.}$, or its equal $\frac{\frac{1}{2}lm + \mathrm{W}'}{m + \mathrm{W}}$ for a in this equation, and we have

$$x^6 - 2b^2 \times \frac{\frac{1}{2}lm + \mathrm{W}'}{m + \mathrm{W}} \times x^3 - \left\{n^2b^2x^2 - (n^2+1)b^4\right\} \times \left(\frac{\frac{1}{2}lm + \mathrm{W}'}{m + \mathrm{W}}\right)^2 = 0.$$

$\text{BA}\cdot\text{BP}\cdot\dfrac{\tan. k}{\tan. k+n} = (\text{BC}+\text{CA})\,(\text{BC}+\text{CP}) \times \dfrac{\tan. k}{\tan. k+n}$ $= \left\{\text{BC}^2 + \text{BC}\,(\text{CA}+\text{CP}) + \text{CA}\cdot\text{CP}\right\} \times \dfrac{\tan. k}{\tan. k+n}$. But $\text{CA} = b\cdot\cot. k$, and $\text{CP} = n\cdot b$; therefore, if y be put $=$ BC, we have

$$x^3 = a\left\{x^2 - \left[y^2 + y\,(b\cot. k + n\cdot b) + \cot. k\cdot nb^2\right] \times \frac{\tan. k}{\tan.\ k+n}\right\}.$$

Exterminating y by means of the equation $x^2 = b^2 + y^2$ we have an equation of the 6*th* power, containing only x and known quantities, and which, when $n = 0$, becomes the ſame as was determined in the laſt caſe.

When the centre of gravity G is placed between the prop and the plane, it is evident that the rod cannot reſt in equilibrio, except ſome other force be oppoſed to the end of the rod ſo as to prevent it from ſliding along the plane. To find what this force muſt be, ſo that the rod may reſt in any aſſigned poſition, we ſhall firſt ſuppoſe the plane to be horizontal, as in fig. 41, pl. 2, and that KB is the rod, making a given angle k with the horizontal plane BD; draw BL and RG perpendicular to BD; and IE, BS perpendicular to BI meeting GR and ID produced in E, P, and S. Then, ſince the directions of the forces acting on the rod at I and G meet in E, it is evident that EB muſt be the direction of a force equivalent to the two forces acting at B in the directions BL and BR; or, if EP repreſent the abſolute force of gravity acting at G then BP will repreſent the preſſure at I; ER the preſſure at B; and BR the force f required.

Becauſe $\text{BR} = \text{GB}\cdot\text{coſ.}\, k$, and $\text{EP} = \text{IS} = \dfrac{\text{IB}}{\text{ſin.}\, k} = \dfrac{\text{ID}}{\text{ſin.}^2 k}$; therefore $\text{BR} : \text{EP} :: \text{GB}\cdot\text{coſ.}\, k : \dfrac{\text{ID}}{\text{ſin.}^2 k}$, or $\text{BR} = \text{EP} \times \dfrac{\text{GB}}{\text{ID}} \times \text{coſ.}\, k\cdot \text{ſin.}^3 k$; that is $f = (m + \text{W}) \times \dfrac{\text{GB}}{\text{ID}} \times \text{coſ.}\, k.\ \text{ſin.}^2 k = \dfrac{\frac{1}{2}lm + \text{W}'}{\text{ID}} \times \text{coſ.}\, k\cdot \text{ſin.}^2 k$.

Again, ſince $\text{BP} = \dfrac{\text{BR}}{\text{ſin.}\, k}$, P is $= \dfrac{\frac{1}{2}lm + \text{W}'}{\text{ID}}\ \text{coſ.}\, k.\ \text{ſin.}\, k$, the preſſure at I; and $\text{ER} = \text{PE} - \text{PR} = \text{PE} - \text{BR} \times \dfrac{\text{coſ.}\, k}{\text{ſin.}\, k}$, therefore $\text{P}' = m + \text{W} - \dfrac{\frac{1}{2}lm + \text{W}'}{\text{ID}} \times \text{coſ.}^2 k.\ \text{ſin.}\, k$. the preſſure at B.

When the weight of the rod only is taken into the account (ſuppoſing it to be of an uniform thickneſs). the problem is then the ſame as the firſt part of the prize queſtion, in the Gentleman's Diary for 1802; and in this caſe $f = \frac{\frac{1}{2}lm}{\text{ID}} \times \text{coſ. } k \cdot \text{ ſin.}^2 k = m \times \frac{l}{2} \times \frac{\text{BD} \cdot \text{DI}}{\text{IB}^3}$; $\text{P} = m \times \frac{l}{2\text{ID}} \times \text{coſ. } k \cdot \text{ ſin. } k$; and $\text{P}' = m \times (\text{ID} - \frac{1}{2}l \text{ coſ.}^2 k \cdot \text{ ſin. } k)$.

If the friction of the plane be equal to n times the preſſure as in the other caſes, take BT : BF :: 1 : n; complete the parallelogram BFTH, and draw EQ parallel to the diagonal HB: then RQ will repreſent the force of friction, and QB the force required. The triangles ERQ, BHF, (or BHT) are evidently ſimilar, and $\text{FB} = n \cdot \text{BT}$; therefore $\text{RQ} = n \text{ RE} = n \cdot \text{P}'$, and $\text{BQ} = \text{BR} - \text{RQ} = f - n \cdot \text{P}'$ the force required.

When BH coincides with BE, Q will coincide with B, and the friction will then be ſufficient to prevent the rod from ſlipping along the plane without the aſſiſtance of any other force.

Again when the plane BD (fig. 42, pl. 2.) makes a given angle with the horizontal plane BV, draw ID and GR perpendicular to the horizon, and draw alſo IE, BS, perpendicular to IB meeting GR and ID produced in E, P, and S. Alſo draw BL and EQ perpendicular to the plane BD and join BE: Then, if EP (or IS) repreſent the force at G, BP will repreſent the force of preſſure at I, EQ the force of preſſure at B, and BQ the force required, acting in the direction BQ.

Put h = the angle DIB, g = the angle IDB, k = the angle DBI, and p = the angle DPV, which the plane makes with the horizon. Then $\text{BR} = \text{GB} \frac{\text{ſin. } k}{\text{ſin. } g}$ and $\text{IS} = \frac{\text{IB}}{\text{coſ. } h}$, therefore $\text{RB} : \text{IS or EP} :: \text{GB} \cdot \frac{\text{ſin. } k}{\text{ſin. } g} : \frac{\text{IB}}{\text{coſ. } h}$, or $\text{BR} = \text{EP} \times \frac{\text{GP}}{\text{BI}} \times \frac{\text{ſin. } h. \text{ coſ. } h}{\text{ſin } g}$; but the angle RBP is the complement of the angle k, and the angle BPR is the complement of the angle h; therefore $\text{PR} = \frac{\text{BR} \times \text{coſ. } k}{\text{coſ. } h}$, and $\text{ER} = \text{EP} - \text{PR} = \text{EP} - \text{BR} \times \frac{\text{coſ. } k}{\text{coſ. } h}$: but $\text{QR} = \text{ER} \cdot \text{ſin. } p$; therefore $\text{QR} = \text{EP} \cdot \text{ ſin. } p$; $- \text{BR} \times \frac{\text{coſ. } k \cdot \text{ ſin. } p}{\text{coſ. } h}$.

Hence

$$\text{Hence BQ} = \text{BR} + \text{QR} = \text{BR} + \left(\text{EP} - \text{BR} \times \frac{\text{coſ.}\,k}{\text{coſ.}\,h}\right).\,\text{ſin.}\,p$$

$$= \text{EP}\left\{\text{ſin.}\,p + \left(1 - \frac{\text{coſ.}\,k.\,\text{ſin.}\,p}{\text{coſ.}\,h}\right).\,\frac{\text{ſin.}\,h.\,\text{coſ.}\,h}{\text{ſin.}\,g}.\,\frac{\text{GR}}{\text{ID}}\right\},$$

$$\text{or } f = (m + \text{W}).\left\{\text{ſin.}\,p + \left(\frac{\text{ſin.}\,h.\,\text{coſ.}\,h}{\text{ſin.}\,g} - \frac{\text{coſ.}\,k.\,\text{ſin.}\,h.\,\text{ſin.}\,p}{\text{ſin.}\,g}.\,\frac{\text{GB}}{\text{IB}}\right\}\right.$$

the force required.

Again $\text{BP} = \text{BR} \times \frac{\text{ſin.}\,g}{\text{coſ.}\,h}$, and $\text{P} = (m + \text{W}) \times \text{ſin.}\,h.\,\frac{\text{GB}}{\text{IB}}$ $= \frac{\frac{1}{2}lm + \text{W}'}{\text{BI}} \times \text{ſin.}\,h$, the preſſure at I.

Alſo $\text{EQ} = \text{ER} \cdot \text{coſ.}\,p = \left(\text{EP} - \text{PR}\,\frac{\text{coſ.}\,k}{\text{coſ.}\,h}\right) \times \text{coſ.}\,p$, and P$'$

$$= (m + \text{W}) \times \text{coſ.}\,p - \frac{\frac{1}{2}lm + \text{W}'}{\text{BI}} \times \frac{\text{ſin.}\,h \cdot \text{coſ.}\,k \cdot \text{coſ.}\,p}{\text{ſin.}\,g},$$

the preſſure at P.

When $p = 0$, $g = 90^\circ$, and ID is put for its equal $\text{IB} \times \frac{\text{ſin.}\,k}{\text{ſin.}\,g}$, the expreſſions for the forces and preſſures, are preciſely the ſame as determined for the horizontal plane.

When the friction of the **plane** is taken into the account, we have only to calculate the values of the forces f and P$'$, as before, and then $f - n\text{P}'$ will be the force required.

SECOND SOLUTION, *by Mr.* CUNLIFFE.

Let the ſtraight line HE (fig. 43, pl. 2.) repreſent the rod, or beam, loaded with the weights; G its centre of gravity; the point P the prop, and EB the vertical plane.

Through P and G draw PB, GT, perpendicular to the vertical plane. When the rod is in equilibrio in the circumſtances mentioned in the queſtion, the centre of gravity G will be at the higheſt point to which it can aſcend: that is, BT will be a maximum in that circumſtance, this is a well known ſtatical principle.

Put $\text{EG} = a$, $\text{PB} = b$, and $\text{PE} = x$; then $\text{GP} = a - x$, and $\text{EB}^2 = x^2 - b^2$, and becauſe PB and GT are parallel,

$$\text{PE}^2 : \text{EB}^2 :: \text{GB}^2 : \text{BP}^2 = \frac{\text{EB}^2 \times \text{GP}^2}{\text{PE}^2} = \frac{(x^2 - b^2)(a - x)^2}{x^2}$$

$=$

$= (a - x)^2 - \frac{b^2 (a - x)^2}{x^2}$ which is to be a maximum. Putting its fluxion $= 0$, and reducing the equation, we find $x = \sqrt[3]{(ab^2)}$.

The problem will be ſomething more general, if enunciated as follows: "To place a ſtraight rod, or beam, loaded with any given weights, ſo that it may be in equilibrio, reſting on an upright prop, and one end touching a ſmooth inclined plane."

Let HE (fig. 44, pl. 2.) repreſent the rod, or beam, loaded with the given weights, and G its centre of gravity; the point P the prop, and EB the ſmooth inclined plane. Through P and G, draw PB, GT, parallel to the horizon, meeting the inclined plane in B and T: and through E draw the vertical line Et meeting PB, GT, produced in b and t; alſo draw PQ perpendicular to EB. When the beam is in equilibrio, in the circumſtances mentioned in the problem, the centre of gravity G will be at the higheſt point to which it can poſſibly arrive; that is, bt, or BT, which has a given ratio to bt, will be a maximum.

Put HE $= a$, PQ $= b$, QB $= d$, and PE $= x$; then PG $= a - x$, EQ $= \sqrt{(x^2 - b^2)}$, and EB $= \sqrt{(x^2 - b^2)} - d$; and by reaſon of the parallels PB, GT; PE : EB :: PG : BT $= \frac{\text{PG} \times \text{EB}}{\text{PE}} = \frac{a - x}{x} \times \left\{ \sqrt{(x^2 - b^2)} - d \right\} = \frac{a\sqrt{(x^2 - b^2)}}{x} - \frac{ad}{x} - \sqrt{(x^2 - b^2)} + d$, which is to be a maximum: putting its fluxion $= 0$, and reducing the reſulting equation, gives $x^3 - ad\sqrt{(x^2 - b^2)} = ab^2$, from which equation x may be found.

When $d = 0$, that is, when the inclined plane becomes vertical, then $x^3 = ab^2$, the ſame as before.

Or, the problem may be reſolved without fluxions.

Let HE (fig. 44, pl. 2.) repreſent the loaded rod; G its centre of gravity; EB the inclined plane, and the point P the prop: draw EF at right angles to EB, meeting GF perpendicular to the horizon in F, and join PF; when the rod is in equilibrio, PF will be perpendicular to GE. For the rod is then ſuſtained by three forces, namely, the force of gravity in the direction GF; the reaction of the inclined plane EB in the direction EF perpendicular thereto; and the preſſure againſt the prop P, which acts in a direction perpendicular to EG at P: and when the rod is in equilibrio, right lines in the ſaid directions meet in the ſame point F.

Draw PQ perpendicular to EB, and draw PB parallel to the horizon, meeting Eb parallel to FG in b. Put GE $= a$, PQ $= b$,

$= b$, $QB = d$, and $EP = x$; then $GP = \;- x$, $EQ = \sqrt{(x^2 - b^2)}$, and $EB = \sqrt{(x^2 - b^2)} - d$. The angles PQE, PbE, being right angles, a circle, whoſe diameter is EP, will paſs through Q and b, and therefore $BP \cdot Bb = QB \cdot EB$; adding PB^2 to each, $PB^2 + BP \cdot Bb = PB \cdot Pb = PB^2 + QB \cdot EB$; whence $Pb = \frac{QB \cdot EB + PB^2}{PB}$; and by the ſimilar triangles PBQ, EBb; $PB : PQ :: EB : Eb = \frac{PQ \cdot EB}{PB}$: alſo, by the ſimilar triangles GPF, EbP, $GP : EF :: Eb = \frac{PQ \cdot EB}{PB} : Pb = \frac{BQ \cdot EB + PB^2}{PB}$, whence $PQ \cdot PF \cdot EB = GP \times (QB \cdot EB + PB^2)$. Again, by the ſimilar triangles PQE, EPF; $PQ : EQ :: EP : PF$; whence $PQ \cdot PF = EP \cdot EQ$, by means of which the preceding expreſſion becomes $EP \cdot EQ \cdot EB = GP \times (QB \cdot EB + PB^2)$; that is, $x\sqrt{(x^2 - b^2)} \times \left\{ \sqrt{(x^2 - b^2)} - d \right\} = (a - x) \times \left\{ d\sqrt{(x^2 - b^2)} + b^2 \right\}$, which equation, when properly reduced, becomes $x^3 - ad\sqrt{(x^2 - b^2)} = ab^2$, which is the very ſame concluſion as derived by the other method of ſolution.

It may be here obſerved, once for all, that the locus of the centre of gravity G is a nodated conchoid, having its axis in the line PQ; this is ſufficiently manifeſt from the generation of that curve.

XXX. PRIZE QUESTION 147, *by* LIMENUS.

Let A and B be two given points without a circle, given in magnitude and poſition; a point C may be found within the circle, ſuch, that if any ſtraight line be drawn through it, cutting the circle in F and G, and AF, BG, be joined, the ſquare of AF will have to the ſquare of BG, the ratio which is compounded of the ratio of FC to CG, and a certain given ratio, and which ratio is alſo to be found?

FIRST SOLUTION, *by* LIMENUS, *the Propoſer.*

Conceive the point C (fig. 45, pl. 2.) to be determined, and let the circle which may paſs through the points F, A, G, meet AC joined in E, then the rectangle $AC \cdot CE$ will be equal to $FC \cdot CG$, which is of a given magnitude, and thus E is a given point.

Alſo, the included triangles ECG, FCA, will be ſimilar, and EG will be to FA as EC to CF; and, likewise, as CG to CA; and therefore EG^2 will be to FA^2 in the ratio compounded of EC to CA, and GC to CF. But, by hypotheſis, AF^2 is to BG^2 in the ratio compounded of FC to CG and a certain given ratio, which let be denoted by that of M to N. And, by compounding theſe two proportions, EG^2 will have to BG^2 the ratio compounded of EC to CA and M to N, which are given ratios. But E and B are given points, and EG and BG are lines drawn from thence to any point in the circumference of a given circle. Therefore thoſe lines, or their ſquares, can only have a given ratio to one another in two circumſtances, namely, when the points E, B, coincide, in which caſe, the ratio of EG^2 to BG^2 is that of equality; and when the other of theſe points E is ſo aſſumed that the given circle may ſatisfy, with reſpect to them, the well known problem of Apollonius on this occaſion. For that purpoſe, the line joining B, E, muſt paſs through the centre O of the given circle, and its radius muſt be a mean proportional to OB, OE; for then EG^2 will always have to BG^2 the duplicate of the ratio in which EB is divided by the circumference, or that of OE to OB. From this we conclude that the point C muſt be ſituated in the line AB or in the line AE, and that it muſt be ſo taken in each caſe, that the points of interſection of any ſtraight line drawn through it, with the given circle, ſhall be in the ſame circle, with the points A, E, or A, B, reſpectively, which are determinate problems.

Take OD in OA, a third proportional to it, and the radius OF; draw BD to meet AE in C; join OC and produce it to K, ſo that the radius OF be a mean proportional to OC, OK, and join KD and KE. As the rectangle BO·OE is equal to AO·OD and to CO·OK, circles may paſs through the points B, E, D, A, and B, E, C, K, and therefore the angle DAE is equal to DBE, and the latter of theſe is equal to CKE; from which it follows that a circle is capable of paſſing through the points O, A, K, E, and that the rectangle EC·CA is conſequently equal to OC·CK. But it may be eaſily ſhewn that the rectangle FC·CG is equal to OC·CK; it is therefore alſo equal to AC·CE, and a circle is capable of paſſing through the points A, F, E, G, and, which, combined with what has been inveſtigated above, demonſtrates that the point C poſſeſſes the required property. And with reſpect to the conſtant ratio of M to N, it has been determined that that compounded with the ratio of EC to CA, muſt be the ſame with that of OE to OB: it is therefore in the ratio compounded of AC to OB and of OE to EC, that is of AK to OK and OK to KB, which is the ſame with the ratio of AK to KB. But KO biſects the angle AKB, and therefore if AB be drawn to meet OC in H, the ratio of M to N will be ſimply as AH to HB.

For the ſecond caſe, draw DE to meet AB in C′, join OC′, and take OK′ a third proportional to OC′ and the radius OF, then by reaſoning in the ſame manner, as in the former caſe, we ſhall perceive that the angle DAK′ is equal to DC′K′, or EC′K′, which is equal to EBK′, and ſo the points O, A, B, K′, are in a circle, and the rectangle AC′·C′B is equal to OC′·C′K′, which is equal to F′C′·C′G′. Thus a circle may paſs through the points A, F′, B, G′, and conſequently C′ is another point where the required property obtains. And from what has been determined, in this caſe, M will be to N as AC′ to BC′. But it is obvious that the interſecting lines are ſo drawn as to determine CH an harmonical mean to CA, CB. Therefore this ratio of AC′ to C′B becomes identicated with the former of AH to HB.

The conſtruction will now be as follows. Find the centre O, of the given circle; join OA, OB, and therein find the points D, E, ſo that OD may be a mean proportional to OA and the radius, and OE a mean proportional to OB and the radius. Join AE, BD, to meet each other in C, through which draw any ſtraight line to meet the circle in F and G; then the ſquare of AF will have to the ſquare of BG the ratio which is compounded of FC to CG and AH to HB. Alſo draw DE to meet AB produced in C′, through which draw any ſtraight line to meet the circle in F′ and G′, and the ſquare of AF′ will have to the ſquare of BG′ the ratio which is compounded of FC′ to C′G′ and AC′ to C′B.

SECOND SOLUTION, *by Mr.* LOWRY.

Suppoſe that the point C (fig. 46, pl. 2.) is found, and that *a* : *b* is the ratio which is to be determined. Then ſince the point C is given (by hyp.) the rectangle GC·CF is alſo given. On the line BC produced take the point D, ſuch, that the rectangle DC·CB may be equal to the given rectangle GC·CF: then D will be a given point, and BC, CD, will be given lines. Join DF; then becauſe the rectangles GC·CF, and DC·CB, are equal, the points B, F, D, G, are in a circle, and the triangles CDF, CGB, are ſimilar;

therefore BG : DF :: CG : CD,
and BG : DF :: BC : CF;
therefore $BG^2 : DF^2 :: CG \cdot BC : CD \cdot CF$:
But, by hypotheſis, $AF^2 : BG^2 :: CF \times a : CG \times b$.
Therefore $AF^2 : DF^2 :: BC \times a : CD \times b$ a given ratio: wherefore AF has to DF a given ratio; and A and D are given points, therefore the locus of the point F will be a circle

given by poſition (Apol. 2, lib. 2nd.). But if this circle meets the given circle in F, then F will be a given point, which is contrary to the hypotheſis. Therefore, in order that the point F may be indeterminate, and that the other conditions of the propoſition may at the ſame time be fulfilled, it is neceſſary that the two circles coincide, or become identical.

When that is the caſe, the line joining the points A and D paſſes through the centre O, and the rectangle OD·OA is equal to the ſquare of the ſemi-diameter of the circle (ibid): and ſince what has been ſhewn with regard to GF is true of any other chord paſſing through C, we have only to determine the point C, ſuch, that the rectangle BC·CD may be equal to the rectangle GC·CF: hence this conſtruction.

Draw AO to the centre of the circle, and take OD a third proportional to AO and the ſemi-diameter, and join BD. Aſſume any point F in the circumference (not in a ſtraight line with the points B, D), and deſcribe a circle through the points B, A, F, cutting the given circle in G; draw GF meeting BD in C: then C is the point ſought.

To determine the ratio $a : b$; draw AI parallel to DC, and produce OC to meet AB in H, and AI in I: then H will be a given point, and AH, BH, given lines; and by the propoſition referred to above

$$\mathrm{AF} : \mathrm{DF} :: \mathrm{OP} \text{ (the radius)} : \mathrm{OD}, \text{ therefore}$$

$\mathrm{AF}^2 : \mathrm{DF}^2 (:: \mathrm{BC} \times a : \mathrm{CD} \times b) :: \mathrm{OP}^2 = \mathrm{OD} \cdot \mathrm{OA} : \mathrm{OD}^2 :: \mathrm{OA} : \mathrm{OD}$, or, $\mathrm{BC} \times a : \mathrm{CD} \times b :: \mathrm{OA} : \mathrm{OD} :: \mathrm{AI} : \mathrm{CD} :: \mathrm{AI} \times b : \mathrm{CD} \times b$; therefore $\mathrm{BC} \times a = \mathrm{AI} \times b$, or $a : b :: \mathrm{AI} : \mathrm{BC} :: \mathrm{AH} : \mathrm{BH}$, the ratio required.

In the preceding analyſis, the point C is within the circle, agreeably to the enunciation of the propoſition; But a point c may be found without the circle, which will poſſeſs properties ſimilar to the point within.

By aſſuming the point c as given, and proceeding exactly as in the former analyſis, we find that AF has to DF a given ratio as before, which can only be the caſe when the ratio is that of equality, and then the points A and D coincide, and the points B, A, c, will be in a ſtraight line. In this caſe $\mathrm{Bc} \times a$ will be equal to $\mathrm{cA} \times b$ ($\mathrm{CD} \times b$), or $a : b :: \mathrm{cA} : \mathrm{cB}$; and the rectangle cA·cB equal to the rectangle cf·cg. We have therefore to determine a point c in AB, ſuch, that drawing any line from c to meet the circle in f and g, the rectangle cf·cg may be equal to the rectangle cA·cB. This is done in the ſame manner as before, by aſſuming any point f in the circumference and deſcribing a circle through the points A, B, f, to interſect the given circle in g, then fg being drawn to meet AB produced in c, it will be the point ſought.

The point C, as determined above, is the ſame as the point O, in queſtion 83, vol. 1, of the Repoſitory. For, if OE be taken a third proportional to BO and the ſemi-diameter, the line joining the points A, E, will paſs through C; alſo, if ED be drawn, it will paſs through c. On OC produced, let ON be taken a third proportional to OC and the ſemi-diameter, and join BN, ND, EC, and AC; then it is obvious that the four points A, D, C, N; N, D, O, B; and N, C, E, B, are in the circumference of the ſame circles; therefore the angle ACN = ADN = ABO = OCE, and the points E, C, A, are in a ſtraight line. Again, draw cO to the centre, and take On a third proportional to Oc and the ſemi-diameter, and join An, Dn, ED, and Dc: then the four points A, D, n, c; A, D, E, B; and A, n, O, B, are alſo in the circumference of the ſame circles; therefore the angle ADc = Anc = ABO = EDO, and the points E, D, c, are in the ſame ſtraight line.

The trapezium A, D, E, B, being inſcribed in a circle, the oppoſite ſides meeting in O and c, and the diagonals in C; it is well known that $a : b$ :: AH : BH, the ſame ratio as when the point C is within the circle.

It is obvious from this analyſis that the given points may be either within or without the given circle, or one of them within and the other without. For if OH meet ED in h, it is evident, from comparing the above proportions, that

$$DF^2 : GE^2 :: CF \times Dh : CG \times Eh,$$

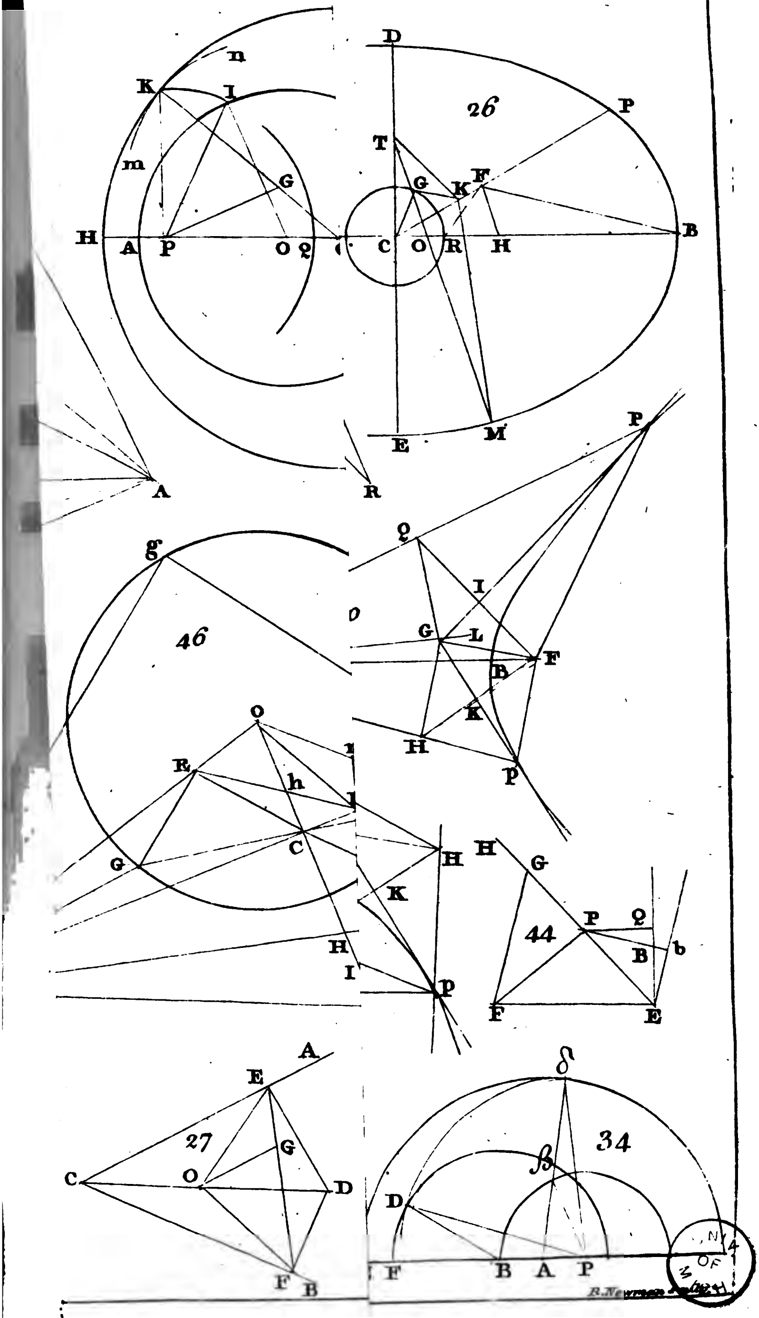
$$DF^2 : BG^2 :: CF \times CD : CG \times CB,$$

$$\text{and } AF^2 : GE^2 :: CF \times CE : CG \times CA.$$

When the points A and B are at the ſame diſtance from the centre, the line EDc will be parallel to AB, and in this caſe the point C only, can ſatisfy the conditions of the propoſition.

The compoſition is ſo evident from the analyſis that its inſertion here is unneceſſary.

Limenus *is requeſted to ſend to Mr.* Glendinning's *for the Medal for ſolving the Prize Queſtion.*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NOTICES

RELATING TO MATHEMATICS.

I. Death of the Bishop of St. Asaph.

Since the publication of our laſt number, the Mathematical Sciences have loſt an able and zealous cultivator, by the death of Dr. Samuel Horſley, Biſhop of St. Aſaph. This venerable prelate is well known to the world, as the author of various works, which have, no doubt, contributed greatly to the intereſts of learning as well as of religion and morality, but it is only of his mathematical writings that we can with propriety ſpeak in this place. In 1769, he publiſhed a Reſtoration of the *Treatiſe of Inclinations of Apollonius*. Our readers no doubt know, that the original has long been loſt. The hiſtorian of the mathematics, *Montucla*, has beſtowed great commendation on this work; he ſays, that the treatiſe of the illuſtrious ancient has been reſtored " in the pure ſtyle, and with the peculiar elegance of the ancient geometry. It is probable, that, could Apollonius return again to the world he would not diſavow it." Dr. Horſley gave to the world, in 1779, an elegant and complete edition of the writings of the immortal Newton, in five volumes quarto. He dedicated this work to the King, and illuſtrated it by a commentary. In 1801, the Biſhop publiſhed an octavo volume, in Engliſh, entitled " Elementary Treatiſes, on the Fundamental Principles of Practical Mathematics, for the uſe of Students. Although publiſhed firſt, this is the laſt in order of three volumes of Elementary Geometry, which he ſent forth from the Clarendon Preſs. The other two volumes are in Latin, the firſt contains the twelve books of Euclid's Elements; and the second, Euclid's Data; A book on the Properties of the Sphere; Archimedes on the Dimenſions of the Circle; and Dr. Keil's Treatiſe on the Nature and Uſe of Logarithms. In 1805, he publiſhed a critical Eſſay on Virgil's two ſeaſons of Honey, and his ſeaſon for ſowing Wheat, with a new and comprehenſive Method of inveſtigating the riſings and ſettings of the fixed Stars. This was the laſt work he publiſhed relating to mathematics. He died in the 69th year of his age.

II.

II. Mathematical Works in the Press.

Mr. Wallace *of the R. M. College, will, in the courſe of the winter, publiſh a ſhort Treatiſe on the Conic Sections.*

A Second Edition of the firſt volume of a Courſe of Mathematics, deſigned for the uſe of the Officers and Cadets, of the Royal Military College by Iſaac Dalby, *Profeſſor of Mathematics in the ſaid College, will ſoon be publiſhed.*

III. Mathematical Works lately Published.

A Treatiſe on Plane and Spherical Trigonometry, with their moſt uſeful practical applications. By John Bonnycaſtle.

A Treatiſe on the Teeth of Wheels, Pinions, &c. demonſtrating the beſt forms which can be given them for the various purpoſes of machinery, ſuch as Clock-work, &c. from the French of M. Camus.

A complete collection of Tables for Navigation, and Nautical Aſtronomy; with ſimple, conciſe, and accurate methods for all the calculations uſeful at Sea. By I. de Mendoza Rios, Esq. F. R. S.

IV. French Books lately Imported.

Traité de Geodeſie, ou Expoſition des methodes Aſtronomiques et Trigonometriques Appliquees ſoit a la meſure de la Terre, ſoit a la confection du Canevas de Cartes et des Plans. Par L. Puiſſant, 4to. *A Paris,* 1805.

Traité Elementaire D'Aſtronomie Phyſique. Par. I. B. Biot. 2 vols. 8vo.

Supplement au dixieme Livre du Traité de Mecanique Celeste ſur L'Action Capillaire. 4to. p p. 65.

Leçons ſur le Calcul dos Fonctions, Nouvelle Edition, revue corrigee et augmentee par l'Auteur. (I. L. Lagrange.) *8vo. A Paris,* 1806.

Note. This work is intended to ſerve as a Commentary and a Supplement to the firſt part of la Theorie des Fonctions Analytiques, *publiſhed in* 1796. *It may be proper to obſerve that the work which we have here noticed was alſo publiſhed in* 1803, *as the* 12*th Cahier of* Journal de L'Ecole Polytechnique.

ARTICLE II.

Solutions to Queſtions propoſed in Number V.

I. QUESTION 151, *by* PAPPUS.

Given the length of a circular arc, to find the length of the chord connecting its extremities, ſo that the area of the included ſegment may be the greateſt poſſible?

FIRST SOLUTION, *by Mr.* W. WALLACE, *R. M. College.*

As the area of the ſegment and the circular arch are each biſected by the diameter, it is evident that an anſwer to this queſtion, and others of the ſame nature, may be eaſily derived from the ſolution of the following ſtill more general problem.

Let *ade* (fig. 47, pl. 3,) be a curve of any kind whatever, given in ſpecies, and having a given poſition with reſpect to a ſtraight line *ac*, to which it is referred as an axis; let ADE (fig. 48, pl. 3,) be another curve ſimilar to *ade*, and AC its axis, and let the ſpaces contained between the curves and their axes be ſimilar to one another. It is required to determine the co-ordinates AB, BD of AD an arch of the latter curve, when that arch is of a given magnitude, and the area ABD contained by the co-ordinates, and the arch is a maximum.

Let the arch *ad* of the curve *ade* be ſimilar to the arch AD, and let *ab*, *bd* be its co-ordinates. Put $ab = x$, $bd = y$, the arch $ad = z$, the given arch $AD = a$, the area $ABD = v$; then, if we conſider that $\int \dot{x}y =$ area *abd*, we have from ſimilar figures,

$$z^2 : a^2 :: \int \dot{x}y : v;$$

hence $v = \frac{a^2 \int \dot{x}y}{z^2}$ a maximum,

and, conſequently, taking the fluxions, &c.

$$\frac{\dot{v}}{\dot{x}} = \frac{a^2}{z^3} \left\{ yz - \frac{\dot{z}}{\dot{x}} \times 2 \int \dot{x}y \right\} = 0;$$

from which we have this general formula,

$$yz = \frac{\dot{z}}{\dot{x}} \times 2\int \dot{x}y;$$

which is applicable, whatever be the nature of the curve.

If we ſuppoſe that ADE is a circle, as in the queſtion, then *ade* being alſo a circle, let us ſuppoſe its radius unity, and put φ for the arch *ad;* then $z = \varphi$, $x = 1 - \text{coſ.}\ \varphi$, $y = \text{ſin.}\ \varphi$, $\dot{z} = \dot{\varphi}$, $\dot{x} = \dot{\varphi}\ \text{ſin.}\ \varphi$, and $2\int \dot{x}y = \varphi - \text{coſ.}\ \varphi\ \text{ſin.}\ \varphi$; hence the general formula $yz = \frac{\dot{z}}{\dot{x}} \times 2\int \dot{x}y$ becomes in this particular caſe,

$$\varphi \times \text{ſin.}\ \varphi = \frac{1}{\text{ſin.}\ \varphi} \times (\varphi - \text{coſ.}\ \varphi\ \text{ſin.}\ \varphi),$$

from which we get,

$$\varphi\ (1 - \text{ſin.}^2\ \varphi) - \text{coſ.}\ \varphi\ \text{ſin.}\ \varphi = 0,$$

that is,

$$\text{coſ.}\ \varphi\ (\varphi\ \text{coſ.}\ \varphi - \text{ſin.}\ \varphi) = 0.$$

Now there are juſt two ways in which we can ſatisfy this equation; we muſt put $\text{coſ.}\ \varphi = 0$, or we muſt put $\varphi\ \text{coſ.}\ \varphi - \text{ſin.}\ \varphi = 0$, which laſt equation is equivalent to $\varphi - \text{tan.}\ \varphi = 0$. If we put $\text{coſ.}\ \varphi = 0$, then we have $\varphi = 90°$; and as to the equation $\varphi - \text{tan.}\ \varphi = 0$, it is evidently impoſſible, ſo long as φ is leſs than two right angles, as it muſt be by the nature of the queſtion; therefore φ can only be a right angle, and the circular ſegment a ſemicircle, and this is the anſwer to the queſtion.

For a ſecond example, let us ſuppoſe the curve ADE, or the curve *ade*, to be a ſemi-parabola, and *ac* its axis. Suppoſe the parameter to be unity; then, becauſe from the nature of the curve $\dot{z} = \dot{x}\sqrt{\left(1 + \frac{1}{4x}\right)}$, we have $\frac{\dot{z}}{\dot{x}} = \sqrt{\left(1 + \frac{1}{4x}\right)}$, and becauſe $2\int \dot{x}y = \frac{4}{3}xy$, the general formula gives in this caſe

$$yz = \tfrac{4}{3}xy \times \sqrt{\left(1 + \frac{1}{4x}\right)},$$

$$\text{and } z = \tfrac{4}{3}x\sqrt{\left(1 + \frac{1}{4x}\right)} = \tfrac{2}{3}\sqrt{(4x^2 + y^2)}.$$

Now if *dt* be drawn touching the curve, and meeting the axis in *d*, it is evident, from the nature of the curve, that the tangent *dt* is equal to $\sqrt{(4x^2 + y^2)}$; therefore in the caſe of the parabola the arch $ad = \frac{2}{3}$ tangent *dt*. And ſince the ſame muſt be true of the curve ADE, if DT be drawn touching it and meeting the axis in T, we muſt alſo have $\text{AD} = \frac{2}{3}\text{DT}$; and ſince AD is by hypotheſis given, therefore DT is given.

Put φ for the angle T, and p for the parameter; then, from the nature of the curve,

$$2 \text{ tangent DT} = p \times \text{cot. } \varphi \text{ coſec. } \varphi,$$

$$\text{arch AD} = \frac{p}{4} \left\{ \text{cot. } \varphi \text{ coſec. } \varphi + \text{hyp. log. } (\text{cot. } \varphi + \text{coſec. } \varphi) \right\}.$$

From theſe equations, and becauſe arch AD $= \frac{2}{3}$ tangent DT, we find

$$\text{cot. } \varphi \text{ coſec. } \varphi = 3 \text{ hyp. log. } (\text{cot. } \varphi + \text{coſec. } \varphi);$$

hence the angle φ may be eaſily determined by the Trigonometrical Tables, and the method of Trial and Error. Now the angle φ, and the line DT being known, the parameter of the parabola, and every thing elſe required, may be readily found.

SECOND SOLUTION, *by Mr.* IVORY, *R. M. College.*

Let a denote the given length of the arc, and let s denote the area of the correſponding ſegment when that area is a maximum, or greater than the area correſponding to an arc of equal length in any other circle. Let z denote the area of the ſegment, that is ſimilar to the ſegment s, in a circle deſcribed with any aſſumed radius (as with the radius 1), and let u be the length of the correſponding arch: then becauſe the ſegments s and z are ſimilar, therefore, as s is to z, ſo is a^2 to u^2; and conſequently, $s = \frac{z}{u^2} \times a^2$.

Let π denote half the circumference, whoſe radius is unity; then, ſuppoſing u to be leſs than π, $z = \frac{u - \text{ſin. } u}{2}$, and $2s = \frac{\pi(u - \text{ſin. } u)}{u^2} \times \frac{a^2}{\pi}$: becauſe an arc is leſs than its tangent, therefore $\frac{2 \text{ſin. } \frac{u}{2}}{\text{coſ. } \frac{u}{2}} > u$; and $2 \text{ ſin. } \frac{u}{2} > u \text{ coſ. } \frac{u}{2}$; and $2 \text{ ſin. } \frac{u}{2} \text{ coſ. } \frac{u}{2}$, or $\text{ſin. } u, > u \text{ coſ.}^2 \frac{u}{2}$; and $u \text{ ſin.}^2 \frac{u}{2} > u - \text{ſin. } u$. But the tangent of the arc $\frac{\pi - u}{2}$ is $= \frac{\text{coſ. } \frac{u}{2}}{\text{sin. } \frac{u}{2}}$; therefore $\frac{2 \text{ coſ. } \frac{u}{2}}{\text{sin. } \frac{u}{2}} > \pi - u$.

Therefore $u\,\text{ſin.}^2\frac{u}{2} \times \frac{2\,\text{coſ.}\frac{u}{2}}{\text{ſin.}\,\frac{u}{2}} > (\pi - u) \times (u - \text{ſin.}\,u)$, that is, $u \times \text{ſin.}\,\frac{u}{2} \times 2\,\text{coſ.}\,\frac{u}{2}$, or $u\,\text{ſin.}\,u, > \pi u - u^2 - \pi\,\text{ſin.}\,u + u\,\text{ſin.}\,u$; and adding $u^2 - u\,\text{ſin.}\,u$ to the unequal quantities, $u^2 > \pi \times (u - \text{ſin.}\,u)$: Therefore when u is less than π, then $2s\left(= \frac{\pi(u - \text{ſin.}\,u)}{u^2} \times \frac{a^2}{\pi}\right)$ is leſs than $\frac{a^2}{\pi}$.

And, if u be ſuppoſed greater than π, then $z = \frac{u + \text{ſin.}\,u}{2}$, and $2s = \frac{\pi(u + \text{ſin.}\,u)}{u^2} \times \frac{a^2}{\pi}$. Let $u = \pi + w$, then, $\text{ſin.}\,u = \text{ſin.}\,w$, and $2s = \frac{\pi(\pi + w + \text{ſin.}\,w)}{(\pi + w)^2} \times \frac{a^2}{\pi}$: but $\pi + w + \text{ſin.}\,w < \pi + 2w$, therefore $2s < \frac{\pi(\pi + 2w)}{(\pi + w)^2} \times \frac{a^2}{\pi}$. Now $\pi(\pi + 2w)$ is leſs than $(\pi + w)^2 = \pi(\pi + 2w) + w^2$; therefore when u is greater than π, $2s$ is leſs than $\frac{a^2}{\pi}$.

But when $u = \pi$, then $2s = \frac{a^2}{\pi}$; and it has been ſhewn that $2s$ is leſs than $\frac{a^2}{\pi}$, both when u is leſs than π, and when it is greater than π: Therefore s is greateſt of all when u is $=$ a ſemicircle. But the ſegment s is ſimilar to the ſegment z; therefore of all the circular ſegments, that have the ſame length of arc, the ſemicircle is the greateſt.

SCHOLIUM.

This curious propoſition is taken from the 5th Book of the Mathematical Collections of Pappus (Prop. 10), and we have done little more than tranſlate the geometrical demonſtration given there into algebraic language. The demonſtration is the more worthy of notice that it is founded on a principle which, although it remained ſteril in the hands of the ancient geometricians, became fertile in conſequences on the diſcovery of the modern methods of Maxima and Minima. If u and a denote ſimilar arcs of two ſimilar curves, and z and s be the correſpondent areas, then becauſe the ſame reaſoning that is applied to ſimilar ſegments of circles in the foregoing demonſtration will equally hold of any ſimilar portions of ſimilar curves, therefore $s = \frac{z}{u^2} \times a^2$. If now, the arch a, be ſuppoſed to be invariable, or of a given length, then it is manifeſt that the area s will be a maximum when $\frac{z}{u^2}$ is a maximum, that is by taking fluxions, when $u \times dz - 2z \times du = 0$. Let x be the abciſs and y the rectangular co-ordinate of the area z, then $dz = ydx$, and $du = dx\sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}$: therefore, by ſubſtitution, $u = \frac{\sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}{y} \times 2\int ydx$. Therefore, "if in any curve whatever an arch u be taken ſo as to ſatisfy the equation $u = \frac{\sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}{y} \times 2\int ydx$, and if another curve be found ſimilar to the propoſed curve, ſuch, that the arch of the ſecond curve, which is ſimilar to the arch u of the firſt curve, ſhall be of a given length a: then the area of the ſecond curve that correſponds to the arch a will be a maximum, that is, it will be greater than the area that correſponds to an arch of the given length a in any other curve ſimilar to the propoſed curve."

In applying this general ſolution to the problem of Pappus, it is to be remarked that z and u are here ſuppoſed to denote the halves of which they were made to ſtand for in the demonſtration above, that is, u denotes half the length of the arch of the required ſegment, and z denotes half the area of it: then $y = \text{ſin.}\ u$, and $x = 1 - \text{coſ.}\ u$, and $\sqrt{\left(1 + \frac{dy^2}{dx^2}\right)} = \sqrt{\left(1 + \frac{\text{coſ.}^2 u}{\text{ſin.}^2 u}\right)} = \frac{1}{\text{ſin.}\ u}$: therefore the formula gives $u = \frac{2\int du \times \text{ſin.}^2 u}{\text{ſin.}^2 u}$ Hence $u\ \text{ſin.}^2 u = 2\int du \times \text{ſin.}^2 u = \int du - \int du \times \text{coſ.}\ 2u = u - \frac{1}{2}\ \text{ſin.}\ 2u$: and $u\ \text{coſ.}^2 u = \frac{1}{2}\ \text{ſin.}\ 2u$, an equation which it is eaſy to prove can not take place unleſs when coſ. u, and ſin. $2u$ are both $= 0$, that is when u is a quadrant, and the whole ſegment is a ſemi-circle.

If the propoſed curve be a parabola, whoſe equation is $4px = y^2$, then the formula gives $u = \frac{\sqrt{(4p^2+y^2)}}{y} \times \frac{1}{3} \times \frac{y^3}{p} = \frac{1}{3} . \frac{y\sqrt{(4p^2+y^2)}}{p}$, but $u = \frac{y\sqrt{(4p^2+y^2)}}{4p} + p \times \text{log.} \frac{y+\sqrt{(4p^2+y^2)}}{2p}$ (Simpſon's fluxions, vol. 1, page 162); therefore, by ſubſtitution, we get $p \times \text{log.} \frac{y+\sqrt{(4p^2+y^2)}}{2p} = \frac{y\sqrt{(4p^2+y^2)}}{12p}$, which equation defines the portion of the curve on which the maximum depends in the caſe of the parabola.

But the principle on which the demonſtration of Pappus is founded, admits of much more extenſive application than to the ſolution of the particular problem we have been conſidering. For if $f(x, y)$, and $F(x, y)$ denote any functions whatever of the co-ordinates of a curve, and if it be required to determine a curve ſimilar to the propoſed curve, ſuch that $f(x, y)$ ſhall be a maximum or minimum, when $F(x, y)$ is of a given magnitude: then we may reſolve this very general problem by reaſoning in the ſame manner as is done above. Let x' and y' denote the co-ordinates of the curve to be found that are ſimilar to the co-ordinates x and y of the propoſed curve, and let m denote the dimenſions of the function $f(x, y)$. (The nature of ſuch queſtions requires that the functions $f(x, y)$ and $F(x, y)$ be homogeneous when the dimenſions of the co-ordinates and the parameters, or conſtant lines that depend on the curve, are reckoned), and let n denote the dimenſions of the function $F(x, y)$: then becauſe x, y, and x', y' are ſimilar co-ordinates of ſimilar curves, by reducing the functions to homogeneous dimenſions, we ſhall get

$$\left\{f(x,y)\right\}^n : \left\{F(x,y)\right\}^m :: \left\{f(x',y')\right\}^n : \left\{F(x',y')\right\}^m$$

whence $f(x', y') = \frac{f(x, y)}{\left\{F(x,y)\right\}^{\frac{m}{n}}} \times \left\{F(x',y')\right\}^{\frac{m}{n}}$

and becauſe $F(x', y')$ is to be of a given magnitude, therefore $f(x', y')$ will be a maximum or minimum, when $\frac{f(x, y)}{\left\{F(x,y)\right\}^{\frac{m}{n}}}$

is a maximum or minimum. And thus the problem is made to depend on a caſe that comes under the common method of maxima and minima.

For the ſake of illuſtration let the following problem be propoſed: Let A (fig. 49, pl. 3,) be a given point in a given horizontal line AD, and BC a line given by poſition in the ſame vertical plane; it is required to determine a cycloid, that ſhall paſs through A and have its baſe in the line AD, ſuch, that a heavy body ſhall deſcend from the point A to the line BC, in that cycloid, in a leſs time than in any other cycloid paſſing through the point A and having its baſe in the line AD.

Let AD be half the baſe of the cycloid to be determined, and DE the diameter of its generating circle: Let PQR (fig. 50, pl. 3,) be the ſemi-cycloid of which the radius of the generating circle is 1, and let PL be the like arch of the ſemi-cycloid PR, that AC is of the ſemi-cycloid AE, and draw the line LM ſimilarly to the line CB: draw LN, LK parallel to QR and PQ, and let LK cut the generating circle in H: let π = the ſemi-circumference QHR, φ = arch QH, and T = the whole time of deſcent in the ſemi-cycloid PLR: then the time of deſcent in the arch PL $= \frac{\varphi}{\pi} \times T$: and the time of deſcent in the arch AC, ſimilar to the arch PL, $= \frac{\varphi}{\pi} \times T \times \frac{\sqrt{(DE)}}{\sqrt{(QR)}}$: but becauſe the lines BC and ML cut the two cycloids ſimilarly, therefore $\frac{DE}{QR} = \frac{AB}{PM}$; therefore the time of deſcent in the arch AC $= \frac{\varphi}{\pi} \times T \times \frac{\sqrt{(AB)}}{\sqrt{(PM)}}$, which will be a minimum when $\frac{\varphi}{\sqrt{(PM)}}$, or $\frac{\text{arch QH}}{\sqrt{(PM)}}$, is a minimum, all the other quantities being conſtant.

I hardly need to remark that $\frac{\text{arch QH}}{\sqrt{(PM)}}$ is what the general formula $\frac{f(x,y)}{\{F(x,y)\}^{\frac{m}{n}}}$ becomes in the particular caſe under conſideration.

Let the cotangent of the angle ABC, or PML, be $= c$; then $NL = QK = 1 - \text{coſ.}\,\varphi$; and $MN = c \times NL, = c - c$

× coſ.

$\times\ \text{coſ.}\ \varphi$: alſo, from the nature of the cycloid $PN =$ arch $QH - KH = \varphi - \text{ſin.}\ \varphi$:
therefore $PM = PN + NM = \varphi - \text{ſin.}\ \varphi + c - c \times \text{coſ.}\ \varphi$.
Therefore $\frac{\varphi}{\sqrt{(PM)}} = \frac{\varphi}{\sqrt{(\varphi - \text{ſin.}\ \varphi + c - c \times \text{coſ.}\ \varphi)}}$, which expreſſion is to be a minimum. By taking fluxions, dividing by $d\varphi$, and throwing away the denominator, there reſults,
$\varphi - \text{ſin.}\ \varphi + c - c \times \text{coſ.}\ \varphi - \frac{1}{2} \times \varphi\,(1 - \text{coſ.}\ \varphi + c \times \text{ſin.}\ \varphi) = 0$.
Hence $\varphi \times \{1 + \text{coſ.}\ \varphi - c \times \text{ſin.}\ \varphi\} = 2\,\{\text{ſin.}\ \varphi - c \times (1 - \text{coſ.}\ \varphi)\}$:
but $1 - \text{coſ.}\ \varphi = \frac{\text{ſin.}^2\ \varphi}{1 + \text{coſ.}\ \varphi}$, therefore, by ſubſtitution,
$\varphi \times (1 + \text{coſ.}\ \varphi) \times \{1 + \text{coſ.}\ \varphi - c \times \text{ſin.}\ \varphi\} = 2\ \text{ſin.}\ \varphi \times \{1 + \text{coſ.}\ \varphi - c \times \text{ſin.}\ \varphi\}$ an equation that cannot take place unleſs $1 + \text{coſ.}\ \varphi = c \times \text{ſin.}\ \varphi$. Now $KR = 1 + \text{coſ.}\ \varphi$ and $KH = \text{ſin.}\ \varphi$, therefore c is the tangent of the angle QRH, and thus we conclude that the chord QH is parallel to LM, and that PM = arc QH, and QM = arch HR = HL. Hence this conſtruction, "Make the angle PQH = the angle ABC, and as arch QH to arch HR ſo make AB to BD; then AD will be the baſe of the ſemi-cycloid required."

James Bernoulli ſeems to be the firſt that, in modern times, obſerved the importance of this principle, drawn from the compariſon of ſimilar curves, in the ſolution of problems relating to maxima and minima. He took the occaſion of his diſcovery to propoſe many problems, relating to circles, parabolas, and cycloids, to his brother John Bernoulli and other rival mathematicians; always taking care to chooſe ſuch inſtances as could be enunciated without any mention of the ſimilarity of the curves under conſideration, that he might not betray the ſource whence he drew his ſolutions (his words are, ne fundamentum ſolutionis nimis aperte darem).

II. QUESTION 152, *by* HERMODORUS.

To inſcribe ſeven equal circles in a given circle?

SOLUTION, *by Mr.* JOHN CAVILL, *Beighton.*

Divide the circumference of the given circle into ſix equal parts in the points A, B, C, D, E and F, (fig. 51, pl. 3,); then the lines AD, BE, and CF which join the oppoſite points of diviſion; will paſs through G, the centre of the circle. With a diſtance equal to a third part of the given radius and centre G, deſcribe a circle to interſect the three diameters in the points *a*, *b*, *c*, *d*, *e*, *f*; then this will be one of the inſcribed circles, and A*a*, B*b*, C*c*, D*d*, E*e*, F*f*, will be the diameters of the ſix other circles.

For since the radius of each of the inſcribed circles is a third part of the radius of the given circle, thoſe circles muſt be equal, and the exterior and interior circles will evidently touch the other ſix; we have therefore only to prove that the circles deſcribed on A*a*, B*b*, C*c*, &c. touch each other.

Let *m* and *n* be the centres of any two contiguous circles, and join *mn*; then ſince AB is a ſixth part of the circumference, the chord AB is equal to the radius AG or GB, and therefore, *mn* is equal to *m*G or *n*G, that is *mn* is equal to *ma* + *nb*, or to the ſum of the radii of the circles deſcribed on the diameters A*a*, B*b*; and, therefore, theſe circles touch each other in the middle of the line *mn*. And it is proved in the ſame manner, that the remaining circles touch each other; and, therefore, ſeven equal circles are inſcribed in the given circle as required.

Solutions were alſo received from Meſſrs. Swale *and* Toplis,

III. QUESTION 153, *by* CENTROBARICOS.

The radius of the greater end of the fruſtum of a paraboloid being R, that of its leſs end *r*, and the height of the fruſtum *h*; it is required to give a neat expreſſion for the poſition of its centre of gravity?

FIRST SOLUTION, *by the Rev.* JOHN TOPLIS, *Arnold*, *Notts.*

Imagine the fruſtum of the paraboloid to be completed, and let *a* denote the part of the axis intercepted between the end (or ordinate) R and the vertex; alſo let *x* denote the thickneſs of the fruſtum from the end R, the poſition of whoſe centre of gravity is required. Then by the properties of the parabola and centre of gravity,

$$\frac{\int x\,\dot{x}\,(a-x)}{\int \dot{x}(a-x)} = \frac{\frac{1}{2}ax^2 - \frac{1}{3}x^3}{ax - \frac{1}{2}x^2} = \frac{3ax - x^2}{6a - 3x}$$

will expreſs the diſtance of the centre of gravity from the end R, meaſured along the axis.

ſtead of being ſpherical, were plane, it is eaſy to ſee by a well known mode of reaſoning that any two ſides, as PR, QR of the inſcribed triangle muſt make equal angles PRB, QRC at the point in which they meet BC a ſide of the circumſcribed triangle; and in like manner it may be ſhewn that the ſame muſt be true when the triangles are ſpherical. Produce RP, RQ, and from A draw A*d*, A*e*, arches of great circles perpendicular to RP, RQ, and A*f* perpendicular to PQ: Then, becauſe the angle APQ, or AP*f* is equal to RPB, that is to AP*d*, it follows that the arch A*f* is equal to the arch A*d*; and in like manner it appears that the arch A*e* is equal to A*f*; thus the arches A*d*, A*e* are equal. Therefore, if a great circle be drawn through A and R, the angle AR*d* is equal to AR*e*; but the angle PRB has been found equal to QRC, therefore the angle ARB is equal to ARC, that is, AR is perpendicular to BC. In the very ſame way it may be ſhewn that the points P and Q are determined by drawing arches of great circles from B and C (the remaining angles of the given triangle) perpendicular to the oppoſite ſides AB, AC; hence the ſolution of the queſtion is manifeſt.

SECOND SOLUTION, *by Mr.* JOHNSON, *the Propoſer.*

Let PQR (fig. 54, pl. 3,) be a triangle inſcribed in the given ſpherical triangle ACB, and having its perimeter the leaſt poſſible. Then, by reaſoning, as in the ſolution to queſtion 120, it will appear that the perimeter of the inſcribed triangle is a minimum, when the angle CQR is equal to the angle AQP, the angle APQ equal to the angle BPR, and the angle BRP equal to the angle CRQ.

To determine the poſition of the points P, Q, R, let the arc CP be drawn to meet QR in I; then the angles CQP, CRP, being the ſupplements of the angles CQR, CRQ, we have by trigonometry

ſin CQ : ſin CP :: ſin CPQ : ſin PQC = ſin CQI
ſin IC : ſin CQ :: ſin CQI : ſin CIQ
therefore ſin IC : ſin CP :: ſin CPQ : ſin CIQ.

It is ſhewn in the ſame manner
that ſin IC : ſin CP :: ſin CPR : ſin CIR
therefore ſin CPQ : ſin CIQ :: ſin CPR : ſin CIR;

but ſin CIQ = ſin CIR, therefore ſin CPQ = ſin CPR, or the angle QPC is equal to the angle CPR; but the angles APQ, BPR are alſo equal, therefore the angle APC is equal to the angle BPC, or CP is perpendicular to AB. In like manner it is proved that the arc AR is perpendicular to CB, and the arc BQ perpendicular to AC, therefore the points P, Q, R are found by drawing perpendiculars from the angular points A, C, B to the oppoſite ſides.

VI. QUESTION 156, *by* LIMENUS.

If a ſtraight line be drawn to meet the four ſides of a trapezium in *a*, *b*, *c*, *d*, and the two diagonals in *m* and *n*; then will *ab* be to *cd* as $am \times nb$ to $cm \times nd$. Required the demonſtration?

SOLUTION, *by* AMICUS.

Fig. 55, pl. 3. Through one of the angular points of the trapezium, as C, draw a ſtraight line parallel to *abcd*, meeting the ſides AB, AC in Q and I, and the diagonal BD in P.

Then by reaſon of the parallels

$$ab : QC :: bn : CP$$

$$CI : cd :: CP : nd$$

$$QC : CI :: am : mc;$$

therefore by compoſition of ratios

$$ab : cd :: am\cdot bn : mc\cdot nd.$$

VII. QUESTION 157, *by* J. W. LEEDS.

Let A be a given point in the periphery of a given circle, let any chord AC be drawn, and divided in the point E, so that the rectangle $AC \times CE$ ſhall be equal to a given ſpace; it is required to determine the equation and quadrature of the locus of the point E?

SOLUTION, *by Mr.* SWALE.

Let AB (fig. 56, pl. 3,) be the diameter of the circle, and take the point Q ſuch, that $AB\cdot BQ$ may be equal to the given rectangle $AC\cdot CE$: draw EP perpendicular to AB and join BC. Put $AB = a$, $BQ = b$, $AQ = c$, any abſciſſa $QP = x$, and the correſponding ordinate $PE = y$.

Then $AE = \sqrt{\{(c-x)^2+y^2\}}$ and by ſimilar triangles

$AE : AP :: AB : AC = \dfrac{a(c-x)}{\sqrt{\{(c-x)^2+y^2\}}}$; and

$AC\cdot CE = AC^2 - AC\cdot AE = AC^2 - AB\cdot AP = ab$;

therefore $\dfrac{a^2(c-x)^2}{(c-x)^2+y^2} - a(c-x) = ab$; and by reduction we have $y = \pm (c - x) \sqrt{\frac{x}{a - x}}$ the equation of the curve.

When y is = to 0, then x is either = to 0, or to c; therefore the curve paſſes through the points Q, A; and if the ordinates be taken on the other ſide of AQ the curve will again paſs through A, therefore A is a *punctum duplex*. Again, if x be greater than c, or P be on BA produced to the left of A, then y is $= \pm (x - c) \sqrt{\frac{x}{a - x}}$, where, if x is equal to a, then y is infinite, which ſhews that if QK be taken = to a, then SKS drawn perpendicular to BK will be an aſymptote to the curve.

When the point Q is between the points A, B, or the given rectangle is leſs than the ſquare of the diameter, the curve has a *nodus* between A and B.

But when the points A and Q coincide, or the given rectangle is equal to the ſquare of the diameter, then $c = 0$, and $y = \pm x \sqrt{\frac{x}{a - x}}$, which is an equation to the ciſſoid of the ancients. In this caſe the nodus vaniſhes, and the curve has a *cuſpis* at A, the two arcs of the curve touching one another in that point.

If the point Q' be on the left of A, or the given rectangle be greater than the ſquare of the diameter, than y is $= \pm (c + x) \sqrt{\frac{x}{a - x}}$, in which equation, if y is $= 0$, then x is alſo $= 0$; and if x is $= a$, then y is infiite, therefore the curve paſſes through Q', and taking Q' K' $= a$, a line drawn through K' perpendicular to AK' will be an aſymptote to the curve, and the nodus will in this caſe become a conjugate point at A.

Laſtly, if we ſuppoſe the point Q'' to be on the right of the point B, then the given rectangle may be either equal to, or greater or leſs than the ſquare of the diameter, and the general equation of the curve will be $y = \pm (c - x) \sqrt{\frac{x}{a - x}}$, where y is $= 0$ if $x = c$, and y infinite when $a = x$; hence the curve paſſes through Q'', and if AK'' be = BQ'', the line S'' K'' S'' perpendicular to AQ will be an aſymptote to the curve.

To find the area we have

$$y\dot{x} = \dot{x}(c - x) \sqrt{\frac{x}{a - x}} = \frac{cx\dot{x}}{\sqrt{(ax - x^2)}} - \frac{x^2\dot{x}}{\sqrt{(ax - x^2)}}.$$

Now $\int\frac{cx\dot{x}}{\sqrt{(ax-x^2)}}$ is $= c\,(z-w)$, where z is the circular arc to radius $\frac{1}{2}a$ and verſed ſine x, and $w = \sqrt{(ax-x^2)}$. Alſo $\int\frac{x^2\dot{x}}{\sqrt{(ax-x^2)}} = \frac{3}{4}\int\frac{ax\dot{x}}{\sqrt{(ax-x^2)}} - \frac{1}{2}xw = \frac{3}{4}az - (\frac{1}{2}x+\frac{3}{4}a)$ w, and $\int y\dot{x} = (c-\frac{3}{4}a)\,z + (\frac{3}{4}a+\frac{1}{2}x-c)\,w =$ (when $x = c$) to $(c-\frac{3}{4}a\ \ z + (\frac{3}{4}a-\frac{1}{2}c)\sqrt{(ac-c^2)}$ = to the whole area of the ſemi-nodus AEQ.

Again, the fluxion of the area Aep is $= \frac{x^2\dot{x}}{\sqrt{(ax-x^2)}} - \frac{cx\dot{x}}{\sqrt{(ax-x^2)}}$ and $\int y\dot{x} = (\frac{3}{4}a-c)\,z + (\frac{1}{2}x-\frac{3}{4}a-c)\,w$ which ought to vaniſh when $x = c$, but it is then $= (\frac{3}{4}a-c)\,z' - (\frac{3}{4}a+\frac{1}{2}c)\,w'$, where z' is the circular arc whoſe radius is $\frac{1}{2}a$, and verſed ſine c, and $w' = \sqrt{(ac-c^2)}$; wherefore the correct fluent is $\frac{3}{4}a-c)(z-z') + (\frac{1}{2}x-\frac{3}{4}a-c)\,w - (\frac{3}{4}a+\frac{1}{2}c)\,w'$ $=$ (when $x = a$) to $(\frac{3}{4}a-c)(z-z') + (c-\frac{1}{4}a)\,w - (\frac{3}{4}a+\frac{1}{2}c)\,w'$ the area of the ſpace contained between the curve and the aſymptote.

Ingenious ſolutions were likewiſe received from Meſſrs. Cavill, Nicholſon, Toplis, *and* Whitley.

VIII. QUESTION 158, *by* JUVENIS.

Given $2x\dot{x} - 3y\dot{x} = 10\dot{y} - 5x\dot{y}$, to find the equation of the fluents?

FIRST SOLUTION, *by the propoſer,* JUVENIS.

In the given equation ſubſtitute $u+2$ for x, and $v+\frac{4}{3}$ for y, (which values are found by MADAM AGNESI's *Analytical Inſtitutions*, Book iv, Sect. ii, Art. 22,) and we ſhall have

$$(2u+4-3v-4)\times\dot{u} = (10-5u-10)\times\dot{v},$$

that is, $2u\dot{u} - 3v\dot{u} = -5u\dot{v}$, or $2u\dot{u} = 3v\dot{u} - 5u\dot{v}$.

Now this equation is of the form of one of thoſe treated of in Art. 7, Sect. i, of the aforeſaid book, and may be ſolved as follows:

Firſt change the ſigns, and it becomes

$$-2u\dot{u} = 5u\dot{v} - 3v\dot{u}.$$

and here c is put inſtead of c^5, which makes no difference, as c is an arbitrary quantity.

THIRD SOLUTION, *by Mr.* CUNLIFFE.

Aſſume $y = A + Bx + v$ where A, B are conſtant quantities, and v is a variable quantity to be determined.

Then the given equation will become

$$2x\dot{x} - 3A\dot{x} - 3Bx\dot{x} - 3v\dot{x} = 10B\dot{x} + 10\dot{v} - 5Bx\dot{x} - 5x\dot{v}.$$

By equating the like terms $-3A = 10B$, or $A = -\frac{10B}{3}$; $2 = -2B$, or $B = -1$; and hence $A = -\frac{10B}{3} = \frac{10}{3}$.

Again, from the equality of the remaining terms of the expreſſion, viz. $-3v\dot{x} = 10\dot{v} - 5x\dot{v}$ we get

$$\frac{\dot{v}}{v} = \frac{3}{5} \times \frac{-\dot{x}}{2 - x};$$

andtaking the fluents

$$\text{hyp. log.}\, v = \text{hyp. log.} \left\{ d \times (2 - x) \right\}^{\frac{3}{5}}, \text{ or } v = \left\{ d \times (2 - x) \right\}^{\frac{3}{5}}.$$

Whence $y = A + Bx + v = \frac{10}{3} - x + \left\{ d \times (2 - x) \right\}^{\frac{3}{5}}$,

where d is a neceſſary correction to be determined from the circumſtances of the problem.

FOURTH SOLUTION, *by* HYPATIA.

From the propoſed equation we have

$$(2x - 3y)\,\dot{x} - (10 - 5x)\,\dot{y} = 0.$$

Put $x = 2 + v$, $y = \frac{4}{3} - z$, then $\dot{x} = \dot{v}$ and $\dot{y} = -\dot{z}$, and the equation becomes

$$(2v + 3z)\,\dot{v} - 5v\dot{z} = 0,$$

which admits of being expreſſed thus,

$$5v\,(\dot{v} - \dot{z}) - 3\,(v - z) = 0.$$

Hence $\frac{5\,(\dot{v} - \dot{z})}{v - z} - \frac{3\dot{v}}{v} = 0$;

therefore $5 \log. (v - z) - 3 \log. v = \log. c$ (c a conſtant given quantity), and $(v - z)^5 - cv^3 = 0$,

therefore $(x - \frac{10}{3} + y)^5 - c\,(x - 2)^3 = 0$

the equation required.

FIFTH

FIFTH SOLUTION, *by* SCOTICUS.

Let the propoſed equation be written thus:

$$\dot{y} + \frac{3y}{10 - 5x}\dot{x} = \frac{2x\dot{x}}{10 - 5x}.$$

Now it is certain from the principles of the fluxional calculus that there is ſome function of x by which if we multiply the expreſſion $\dot{y} + \frac{3y}{10 - 5x}\dot{x}$ the product ſhall be a complete fluxion; let this function be X; therefore $X\dot{y} + \frac{3Xy}{10 - 5x}\dot{x}$ muſt be a complete fluxion. But is known, that when any expreſſion of this form $M\dot{x} + N\dot{y}$ (where M and N denote any functions of x and y) is a complete fluxion, then the fluxion of M taken upon the ſuppoſition that y only is variable, and divided by $\dot{y}$ is equal to the fluxion of N, taken upon the ſuppoſition that x only is variable, and divided by $\dot{x}$. In the preſent caſe $M = \frac{3Xy}{10 - 5x}$, $N = X$, hence $\frac{\dot{M}}{\dot{y}} = \frac{3X}{10 - 5x}$, $\frac{\dot{N}}{\dot{x}} = \frac{\dot{X}}{\dot{x}}$, therefore $\frac{3X}{10 - 5x} = \frac{\dot{X}}{\dot{x}}$, and $\frac{\dot{X}}{X} = \frac{3}{5} \cdot \frac{\dot{x}}{2 - x}$, and $X = (2 - x)^{-\frac{3}{5}}$, therefore the propoſed equation when prepared for integration is

$$(2 - x)^{-\frac{3}{5}}\dot{y} + \frac{3}{5}(2 - x)^{-\frac{8}{5}} y\dot{x} = \frac{2}{5}x\dot{x}(2 - x)^{-\frac{8}{5}};$$

and the equation of the fluents is

$$(2 - x)^{-\frac{3}{5}}y = \frac{4}{3}(2 - x)^{-\frac{3}{5}} + (2 - x)^{\frac{2}{5}} + c,$$

which by proper reduction becomes

$$\frac{(3y + 3x - 10)^5}{(2 - x)^3} = \text{a conſtant quantity.}$$

SIXTH SOLUTION, *by Mr.* IVORY.

Let the equation be brought to this form

$$\dot{y} + y \times \frac{3}{5} \cdot \frac{\dot{x}}{2 - x} = \frac{2}{5} \cdot \frac{x\dot{x}}{2 - x}:$$

aſſume

from B, let any ſtraight line BDQ be drawn meeting the circle in D and the tangent in Q: from D, along DB, ſet off DP equal to QD, and D will be a point in the curve.

Neat Solutions were alſo received from Meſſrs. Cavill, Nicholſon, Swale *and* Toplis.

X. QUESTION 160, *by Mr.* CUNLIFFE, *R. M. College.*

Required the ſum of the infinite ſeries

$$1-\tfrac{1}{2}\times(\tfrac{1}{3})+\tfrac{1}{3}\times(\tfrac{1}{3}+\tfrac{1}{5})-\tfrac{1}{4}\times(\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7})+\tfrac{1}{5}\times(\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{9})-\&c.$$

FIRST SOLUTION, *by Mr.* WALLACE.

Let $x^2-\tfrac{1}{3}\cdot\frac{x^4}{2}+(\tfrac{1}{3}+\tfrac{1}{5})\frac{x^6}{3}-(\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7})\frac{x^8}{4}+(\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{9})\frac{x^{10}}{5}$ — &c. be the expanſion of y, a function of a variable quantity x: the ſum of the propoſed ſeries is evidently the value of y correſponding to $x = 1$; therefore, to find that ſum we muſt inveſtigate the function from the above expreſſion for its expanſion.

From the equation

$$y = x^2 - \tfrac{1}{3}\cdot\frac{x^4}{2} + (\tfrac{1}{3}+\tfrac{1}{5})\frac{x^6}{3} - (\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7})\frac{x^8}{4} + \&c.$$

by taking the fluxions and dividing by $2\dot{x}$ we have

$$\frac{\dot{y}}{2\dot{x}} = x - \tfrac{1}{3}x^3 + (\tfrac{1}{3}+\tfrac{1}{5})\,x_5 - (\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7})\,x^7 + \&c.$$

Let the firſt term of the ſeries be tranſpoſed to the other ſide of the equation, and let the whole be multiplied by x^2, thus we have

$$\frac{\dot{y}x^2}{2x} - x^3 = -\tfrac{1}{3}x^5 + (\tfrac{1}{3}+\tfrac{1}{5})\,x^7 - \&c.$$

Let each ſide of this laſt equation be added to the correſponding ſide of the preceding equation and the reſult is

$$\frac{\dot{y}}{2\dot{x}}\,(1+x^2) - x^3 = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \&c.$$

Therefore

$$\dot{y} = \frac{2\dot{x}x^3}{1+x^2} + \frac{2\dot{x}}{1+x^2}\left\{x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \&c.\right\}$$

Now if φ denote the arch whoſe tangent is x, (radius being unity)

unity) then it is known that $\varphi = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \text{\&c.}$ and $\dot{\varphi} = \frac{\dot{x}}{1 + x^2}$, therefore

$$\dot{y} = \frac{2\dot{x}x^3}{1 + x^2} + 2\dot{\varphi}\varphi = 2x\dot{x} - \frac{2x\dot{x}}{1 + x^2} + 2\dot{\varphi}\varphi$$

and taking the fluents, ſo that $y = 0$ when $x = 0$, $y = x^2 - \text{hyp. log. } (1 + x^2) + \varphi^2$.

But when $x = 1$, then $\varphi = \frac{1}{4}\pi$; (here $\pi = 3{\cdot}14159$) therefore the ſum of the propoſed ſeries is $1 - \text{h. log. } 2 + \frac{1}{16}\pi^2$.

Scholium. In the ſame manner we may find the ſum of the ſeries

$$a - \left(\frac{1}{m}\right)\frac{1}{n} + \left(\frac{1}{m} + \frac{1}{m+i}\right)\frac{1}{n+d} - \left(\frac{1}{m} + \frac{1}{m+i} + \frac{1}{m+2i}\right)\frac{1}{m+2d} +$$

&c. which may be conſidered as the value of this other ſeries

$$a - \left(\frac{1}{m}\right)\frac{x^{in}}{n} + \left(\frac{1}{m} + \frac{1}{m+i}\right)\frac{x^{i(n+d)}}{n+d} - \left(\frac{1}{m} + \frac{1}{m+i} + \frac{1}{m+2i}\right)\frac{x^{i(n+2d}}{n+2d}$$

+ &c. when x is ſuppoſed $= 1$. Let y denote the function which being expanded produces this laſt ſeries, then, proceeding in all reſpects as before, we find

$$\dot{y} = \frac{i\dot{x}}{1 + x^{id}}\left\{- \frac{x^{in-1}}{m} + \frac{x^{i(n+d)-1}}{m+i} - \frac{x^{i(n+2d)-1}}{m+2i} + \text{\&c.}\right\}$$

But the ſeries in the parentheſis is the expanſion of

$$- dx^{in - dm - 1} \times \int \frac{\dot{x}x^{md-1}}{1 + x^{di}},$$

Let this expreſſion be denoted by X, then

$$y = \int \frac{ixX}{1 + x^{id}}$$

the fluent being ſo taken that $\dot{x} = a$ when $x = 0$.

SECOND SOLUTION, *by Mr.* CUNLIFFE, *the Propoſer.*

Aſſume $\int \frac{\dot{x}}{1 + x^2} \times (x - ax^3 + bx^5 - cx^7 + dx^9 - cx^{11} + \text{\&c.})$

$$= \frac{x^2}{2} - \tfrac{1}{4}Px^4 + \tfrac{1}{6}Qx^6 - \tfrac{1}{8}Rx^8 + \tfrac{1}{10}Sx^{10} - \tfrac{1}{12}Tx^{12} \text{ \&c.}$$

taking the fluxions and dividing by $\frac{\dot{x}}{1+x^2}$

$$x-ax^3+bx^5-cx^7+dx^9\ \&c. = (1+x^2)\times(x-Px^3+Qx^5-Rx^7\ \&c.)$$
$$= x - Px^3 + Qx^5 - Rx^7 + Sx^9 - Tx^{11}\ \&c.$$
$$+\ x^3 - Px^5 + Qx^7 - Rx^9 + Sx^{11}\ \&c.$$

making the coefficients of the like terms equal to each other, $P - 1 = a$ or $P = 1 + a$; $Q - P = b$, or $Q = P + b = 1 + a + b$; $R - Q = c$, or $R = Q + c = 1 + a + b + c$ &c. therefore

$$\int\frac{\dot{x}}{1+x^2}\times(x - ax^3 + bx^5 - cx^7 + dx^9 - ex^{11}\ \&c.) =$$
$$\frac{x^2}{2} - \tfrac{1}{4}x^4(1+a) + \tfrac{1}{6}x^6(1+a+b) - \tfrac{1}{8}x^8(1+a+b+c) + \tfrac{1}{10}x^{10}$$
$$(1+a+b+c+d)\ \&c.$$

Take $a = \frac{1}{3}$, $b = \frac{1}{5}$, $c = \frac{1}{7}$, &c. then $x - ax^3 + bx^5 - cx^7$ &c. $= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$ &c. $= \int\frac{\dot{x}}{1+x^2} = z$, where z denotes the length of a circular arc, radius 1 and tangent x.

Therefore $\int\frac{\dot{x}}{1+x^2}\times(x - ax^3 + bx^5 - cx^7 + dx^9 - ex^{11}$ &c.

$$= \frac{z^2}{2} = \frac{x^2}{2} - \tfrac{1}{4}Px^4 + \tfrac{1}{6}Qx^6 - \tfrac{1}{8}Rx^8 + \tfrac{1}{10}Sx^{10} - \tfrac{1}{12}Tx^{12}\ \&c.$$

or $z^2 = x^2 - \frac{1}{2}Px^4 + \frac{1}{3}Qx^6 - \frac{1}{4}Rx^8 + \frac{1}{5}Sx^{10} - \frac{1}{6}Tx^{12}$ &c.

$$= x^2 - \tfrac{1}{2}x^4(1+\tfrac{1}{3}) + \tfrac{1}{3}x^6(1+\tfrac{1}{3}+\tfrac{1}{5}) - \tfrac{1}{4}x^8(1+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7})\ \&c.$$

Take $x = 1$, then $\frac{q^2}{4} = 1 - \frac{1}{2}(1+\frac{1}{3}) + \frac{1}{3}(1+\frac{1}{3}+\frac{1}{5}) - \frac{1}{4}(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}) + \frac{1}{5}(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}) - \frac{1}{6}(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11})$ &c. where q denotes the length of the quadrantal arc of a circle radius 1.

And therefore $1 + \frac{q^2}{4} = 2 - \frac{1}{2}(1+\frac{1}{3}) + \frac{1}{3}(1+\frac{1}{3}+\frac{1}{5}) - \frac{1}{4}(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7})$ &c.

alſo h. l. (2) $= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$ &c.

and taking the difference

$$1 + \frac{q^2}{4} - \text{h. l.}\,(2) = 1 - \tfrac{1}{2}(\tfrac{1}{3}) + \tfrac{1}{3}(\tfrac{1}{3}+\tfrac{1}{5}) - \tfrac{1}{4}(\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}\ \&c.$$

ad infinitum.

THIRD

THIRD SOLUTION, *by Mr.* IVORY.

Put $S = z^2 - \frac{1}{3} \cdot \frac{z^4}{2} + (\frac{1}{3} + \frac{1}{5}) \frac{z^6}{3} - (\frac{1}{3} + \frac{1}{5} + \frac{1}{7}) \frac{z^8}{4} +$ &c.

And, by taking the fluxions,

$$\tfrac{1}{2} \times \frac{dS}{dz} = z - \tfrac{1}{3} \cdot z^3 + (\tfrac{1}{3} + \tfrac{1}{5})\, z^5 - (\tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7})\, z^7 + \&c.$$

And, by multiplying by $1 + z^2$,

$$\tfrac{1}{2} \times \frac{dS}{dz} \times (1 + z^2) = z^3 + z - \tfrac{1}{3}z^3 + \tfrac{1}{5}z^5 - \tfrac{1}{7}z^7 + \&c.$$

Let $z =$ tangent φ, then $\varphi = z - \frac{1}{3}z^3 + \frac{1}{5}z^5 - \frac{1}{7}z^7 +$ &c.

And, by ſubſtitution, $\frac{1}{2} \times \frac{dS}{dz} \times (1 + z^2) = z^3 + \varphi$:

And, by multiplying by $\frac{2dz}{1 + z^2} = 2d\varphi$,

$$dS = \frac{2z^3dz}{1 + z^2} + 2\varphi d\varphi :$$

And by integrating $S = z^2 - \log.\ (1 + z^2) + \varphi^2$, no conſtant quantity being neceſſary. Let z be taken $= 1$, then the ſum of the ſeries propoſed to be ſummed $= 1 - \log.\ 2 + \frac{\pi^2}{16}$, where $\pi = 3{\cdot}14159$ &c.

Mr. Cavill *alſo anſwered it.*

XI. QUESTION 161, *by Mr.* W. SIMPSON, *Bolton.*

Let ACB be a given triangle, D, L, Q, given points in the ſide AB; through D draw any right line DEF, cutting AC, BC, in E and F reſpectively, and through E and F draw right lines LEP, QFP interſecting in P. Required the locus of P?

FIRST SOLUTION, *by Mr.* WALLACE.

Fig. 58, pl. 3. Through C draw RCH parallel to AB; produce DE, QF, LE to meet RH in H, K, and R, and join CP meeting AB in G.

The triangles HFC, DFB, have their baſes HC, BD, ſimilarly divided at K and Q, therefore HC : KC :: DB : BQ; but DB and BQ being given lines, the ratio of DB to BQ is given; therefore the ratio of HC to KC is given. Again, the triangles HER, DEL, have their baſes HR, DL, ſimilarly divided at C and A, therefore HC : CR :: DA : AL; now the ratio of DA to AL is given, for DA and AL are given, therefore the ratio of HC to CR is given; but it has been ſhewn that the ratio of HC to CK is given; therefore the ratio of CK to CR is given. Now the triangles QPL, RPK have their baſes QL, RK, ſimilarly divided at G and C, therefore QG : GL :: KC : CR; but the ratio of KC to CR has been ſhewn to be given, therefore the ratio of QG to GL is given; and Q and L being by hypotheſis given points, the point G is given, and the line CG is given by poſition. Therefore the locus of the point P is a ſtraight line CG given by poſition.

SECOND SOLUTION, *by Mr.* LOWRY, *R. M. College.*

Fig. 59, pl. 3. Draw CP to meet AB in K, and through P draw GO parallel to AB, meeting DE in G, AC in H, and BC in O.

Then by ſimilar triangles HP : GP :: AL : DL,
and GP : PO :: DQ : BQ;
therefore by compounding HP : PO :: AL·DQ : DL·BQ, which is a given ratio, becauſe AL, DQ, DL, and BQ are given lines; and ſince HO is parallel to the baſe AB, and is divided at P in a given ratio, the locus of the point C is a ſtraight line paſſing through C, and dividing the baſe AB into two parts AK, BK, which have the ſame given ratio that HP has to PO.

The ſame algebraically.

Put DL $= a$, QB $= b$, DQ $= c$, AL $= d$, AB $= m$, AC $= n$, HC $= x$, and PH $= y$; then by ſimilar triangles

$$\mathrm{AL} = d : \mathrm{PH} = y :: \mathrm{LD} = a : \mathrm{PG} = \frac{ay}{d},$$

$$\text{and } \mathrm{DQ} = c : \mathrm{PG} = \frac{ay}{d} :: \mathrm{QB} = b : \mathrm{PO} = \frac{aby}{cd};$$

$$\text{Hence } \mathrm{HO} = y + \frac{aby}{cd} = \frac{cd + ab}{cd} \times y.$$

$$\text{And } \mathrm{AB} = m : \mathrm{AC} = n :: \mathrm{HO} : \mathrm{HC} = \frac{cd + ab}{mcd} \times ny,$$

or $x = \dfrac{cd + ab}{mcd} \times ny$, an equation to a ſtraight line.

When $x = 0$, y is $= 0$, and the locus paſſes through C; alſo when $x = n$, y is $= \frac{mcd}{cd + ab}$; hence if AK be taken $=$ to $\frac{mcd}{cd + ab}$, the line CK will be the locus required.

XII. QUESTION 162, *by* AMICUS.

If a given weight be ſuſtained by means of a ſtring whoſe ends are connected (by loops or rings) to two rods placed in the ſame vertical plane, and making given angles with the horizon; it is required to aſcertain the poſition of the ſaid weight when it reſts in equilibrio, ſuppoſing it has liberty to ſlide freely along the ſtring?

SOLUTION, *by* AMICUS, *the Propoſer.*

Fig, 60, pl. 3. Let P and Q be the points where the rings preſs upon the rods CA, CB; and W the poſition of the weight when it is at reſt. Draw AB parallel to the horizon, and WG perpendicular to AB. Then ſince the rings and weights are to be ſtationary, the ſtrings PW, QW muſt be perpendicular to the rods; for if they were in any other directions as W*p*, W*q*, then the forces acting in thoſe oblique directions might be each decompoſed into two others, *p*P, PW, and Q*q*, QW; and as there is no force to counteract the effect of either of the forces, *p*P, Q*q*, which act in the direction of the rods, the rings would have a tendency to move in thoſe directions till they came to the points P, Q, where the ſtrings are perpendicular to the rods. Alſo, ſince the weight W is at liberty to ſlide freely along the ſtring, the forces acting in the direction of the ſtrings WP, WQ muſt be equal; for if the one force was greater than the other, the weight would have a tendency to move towards that ſide to which it was drawn by the greater force.

Now when the forces in the directions PW, QW are equal, their reſultant (which is directly oppoſed to the action of gravity on the weight W) will biſect the angle PWQ, or the angle PWG will be equal to the angle QWG; and, therefore, ſince GW is perpendicular to AB, the angle AWP will be equal to the angle BWP, which (becauſe the angles at P and Q are right angles) can only be the caſe when the angle WAP is equal to the angle WBQ, or when the rods are equally inclined to the horizon.

In every other caſe the weight will have a tendency to ſlip along the ſtring till it comes to one of the rings, and then the ſtring will be perpendicular to the other rod.

To find the poſition of the weight, when the rods are equally inclined to the horizon, draw AD perpendicular to CB, meeting WE, a parallel to CB, in E; then ſince the angles DAB, QWB, AWP are equal, the right-angled triangles APW, AEW are equal in all reſpects; and therefore AE is = to PW; and ED is = to WQ by reaſon of the parallels; therefore AD is = to PW + WQ, or the length of the ſtring. Hence it is evident that if AD be drawn perpendicular to CB, ſo as to be equal to the length of the ſtring, and AB be drawn parallel to the horizon, or CB be taken equal to CA; then the point W may be any where in the line AB; for if the lines WP, WQ, be perpendicular to the rods they will make equal angles with AB, and their ſum will always be equal to the length of the ſtring.

The preſſure of the rings on each rod is equal to the weight W multiplied by the ſine of the angle CAB.

A ſimilar reſult may be obtained by finding the locus of the point W, without taking into conſideration the equality of the forces acting in the direction of the ſtrings. For ſince the ſum of the perpendiculars WP, WQ, is equal to a given line, the locus of the point W is a ſtraight line AB given by poſition, as is pretty generally known; and as the weight can only be at reſt when it has deſcended as low as it poſſibly can; it is evident that when the locus AB is parallel to the horizon, the weight will reſt at any aſſigned point of the line AB; but if it be inclined to the horizon, then the weight will continue to deſcend till it comes to the loweſt point of the locus which will be at one of the rings.

If one end of the thread was faſtened at any point Q in the rod BC, while the other end was at liberty to ſlip along AC, the ſtring PW would ſtill be perpendicular to CA, and the line WG would biſect the angle PWQ, as before; and therefore the angle BWQ would be equal to the angle PWA, or to the complement of the given angle WAP; and ſince the angle WBQ is given, the angle WQB would alſo be given: and the poſition of the point W might be determined by drawing QW to make the given angle with BC, and taking QI = to the length of the ſtring; then PI, drawn perpendicular to the horizon, will meet AC in the point P; and PW, drawn perpendicular to AC, will meet IQ in W, the point required. For it is evident that the triangle IWP is iſoſceles, and therefore PW + WQ = IQ = to the length of the ſtring.

XIII. QUESTION 163, *by Mr.* CUNLIFFE,

Find ſuch poſitive values of x, y, and z, in whole numbers, as will make the formulæ

$x^2 + xy + y^2$, $x^2 + xz + z^2$ and $y^2 + yz + z^2$,

all rational ſquares?

SOLUTION, *by Mr.* CUNLIFFE, *the Propoſer.*

Aſſume $x + y - a$ for the root of the firſt, and $x + z - b$ for the root of the ſecond;

then $x^2 + xy + y^2 = (x + y - a)^2 = x^2 + 2xy + y^2 - 2a(x + y) + a^2$,

and $x^2 + xz + z^2 = (x + z - b)^2 = x^2 + 2xz + z^2 - 2b(x + z) + b^2$;

from the firſt of which $x = \dfrac{2ay - a^2}{y - 2a}$;

and from the ſecond $x = \dfrac{2bz - b^2}{z - 2b}$;

therefore $\dfrac{2ay - a^2}{y - 2a} = \dfrac{2bz - b^2}{z - 2b}$: whence $y = \dfrac{z(a^2 - 4ab) - 2ab(a - b)}{2z(a - b) - 4ab + b^2}$;

$$y + z = \frac{z(a^2 - 4ab) - 2ab(a - b)}{2z(a - b) - 4ab + b^2} + z$$

$= \dfrac{2z^2(a - b) + z(a^2 - 8ab + b^2) - 2ab(a - b)}{2z(a - b) - 4ab + b^2}$, and by means of what has been deduced $y^2 + yz + z^2 = (y + z)^2 - yz$

$$= \frac{\{2z^2(a - b) + z(a^2 - 8ab + b^2) - 2ab(a - b)\}^2}{\{2z(a - b) - 4ab + b^2\}^2}$$

$$- \frac{\{z^2(a^2 - 4ab) - 2abz(a - b)\} \times \{2z(a - b) - 4ab + b^2\}}{\{2z(a - b) - 4ab + b^2\}^2} = \text{a}$$

ſquare; therefore $\{2z^2(a - b) + z(a^2 - 8ab + b^2(- 2ab(a - b)\}^2 - \{z^2(a^2 - 4ab) - 2abz(a - b)\} \times \{2z(a - b) - 4ab + b^2)\} = 4z^4(a - b)^2 + 2z^3(a - b)(a^2 - 12ab + 2b^2) + z^2(a^4 - 16a^3b + 57a^2b^2 - 16ab^3 + b^4) - 2abz(a - b)(2a^2 - 12ab + b^2) + 4a^2b^2(a - b)^2$ = a ſquare.

Aſſume $2z^2(a-b)+z\left(\frac{a^2-12ab+2b^2}{2}\right)-2ab(a-b)$ for its root: Then $4z^4(a-b)^2+2z^3(a-b)(a^2-12ab+2b^2)+z^2(a^4-16a^3b+57a^2b^2-16ab^3+b^4)-2abz(a-b)(2a^2-12ab+b^2)+4a^2b^2(a-b)^2=\left\{2z^2(a-b)+z\left(\frac{a^2-12ab+2b^2}{2}\right)-2ab(a-b)\right\}^2=4z^4(a-b)^2+2z^3(a-b)(a^2-12ab+2b^2)+\frac{z^2(a^2-12ab+2b^2)^2}{4}-z^2\times 8ab(a-b)^2-2abz(a-b)(a^2-12ab+2b^2)+4a^2b^2(a-b)^2$, which after proper reduction gives $\frac{8b(a-b)(a^2-b^2)}{3a^3-8a^2b+16ab^2+16b^3}$, where a and b may be taken at pleaſure.

Example 1. Take $a=3$, and $b=2$, then $z=\frac{80}{257}$, $x=\frac{b^2-2bz}{2b-z}=\frac{59}{79}$, and $y=\frac{a^2-2ax}{2a-x}=\frac{357}{415}$; which fractions being reduced to a common denominator, and that denominator rejected will give $z=80\times 79\times 415=2622800$; $y=357\times 257\times 79=7248171$ and $x=59\times 257\times 415=6292645$, which are three numbers that will anſwer.

Example 2. Take $a=4$, and $b=3$, then $z=\frac{7}{34}$, $x=\frac{264}{197}$, and $y=\frac{65}{72}$, and ordering theſe fractions as in the firſt example we may take $z=7\times 197\times 72$, $x=264\times 34\times 72$, and $y=65\times 197\times 34$, or the halves of theſe numbers will do, *viz.* $x=264\times 34\times 36=323136$, $y=65\times 197\times 17=217685$, and $z=7\times 197\times 36=49644$, which numbers will anſwer, and are ſomething ſmaller than the former ſet.

Again, aſſume $2z^2(a-b)+\frac{z}{2}(a^2-12ab+2b^2)+2ab(a-b)$ for the root of the general expreſſion which is to be made a ſquare, that is, put $4z^4(a-b)^2+2z^3(a-b)(a^2-12ab+2b^2)+z^2(a^4-16a^3b+57a^2b^2-16ab^3+b^4)-2abz(a-b)(2a^2-12ab+b^2)+4a^2b^2(a-b)^2=\left\{2z^2(a-b)+\frac{z(a^2-12ab+2b^2}{2}+2ab(a-b)\right\}^2=4z^4(a-b)^2+2z^3(a-b)(a^2-12ab+2b^2)+\frac{z^2(a^2-12ab+2b^2)^2}{4}$

$+ z^2 \times 8ab\ (a - b)^2 + 2abz\ (a - b)\ (a^2 - 12ab + 2b^2) + 4a^2b^2\ (a - b)^2$, which after proper reduction gives $z = \frac{8b\ (a - b)\ (a^2 - 8ab + b^2)}{a^3 - 24a^2b + 48ab^2 - 16b^3}$ which is a ſecond formula for the value of z.

Example 1. Take $b = 2$ and $a = 1$, then $z = \frac{176}{17}$, $x = \frac{53}{9}$, and $y = \frac{97}{35}$; and ordering theſe fractions as in the former examples we may take $z = 176 \times 9 \times 35 = 55440$, $x = 53 \times 17 \times 35 = 31535$, and $y = 97 \times 9 \times 17 = 14841$; and theſe numbers are ſtill leſs than either of the preceding ſets.

Example 2. Take $b = 7$ and $a = 4$; then $z = \frac{7 \times 53}{18}$, $x = \frac{616}{17}$ and $y = \frac{97}{10}$; and proceeding with theſe fractions as before we ſhall have, after dividing the numbers by their greateſt common meaſure, $z = 371 \times 17 \times 5 = 31535$, $x = 616 \times 9 \times 10 = 55440$, and $y = 97 \times 17 \times 9 = 14841$ which are the very numbers found in the preceding example of the preſent formula.

XIV. QUESTION 164, *by* URSA MINOR.

A ſtraight line drawn through the focus of any conic ſection, to the point in which a diameter meets the directrix, will be perpendicular to a tangent at either extremity of that diameter?

FIRST SOLUTION, *by* AMICUS.

Fig. 61, pl. 3. Let the diameter DC meet the directrix AQ in A, and draw AF to the focus F; then AF will be perpendicular to GH, a tangent at either extremity of the diameter.

When the ſection is an ellipſe, or a hyperbola, let *f* be the other focus; join *f*C, FC, and DF, and draw CP, DQ perpendicular to the directrix. Let the tangent GC meet AF in H, and DF produced in E; then the triangles DFB, *f*CB, having two ſides of the one equal to two ſides of the other, and the angles at B equal, they are equal in all reſpects: therefore the angle *f*CD is equal to the angle CDF; and *f*C is parallel to DE;

therefore the angle DEC is equal to the angle GC*f*, or to the angle FCE, which is equal to the angle GC*f*, as is proved by every writer on conic ſections.

Again, by a general property of the ſections DQ : CP :: FD : CF :: AB : AC, therefore (Eu. vi. prop. A,) AF biſects the angle CFE, and the triangles CFH, EFH having the angle HCF equal to the angle HEF, and the angle CFH to the angle EFH, have alſo the angle CHF equal to the angle EHF; and therefore AF is perpendicular to the tangent GH. It is obvious that AF is alſo perpendicular to the tangent at the other extremity of the diameter: for ſince tangents at the extremities of any diameter of an ellipſe or hyperbola are parallel; a line which is perpendicular to one of the tangents will alſo be perpendicular to the other.

When the curve is a parabola. Fig. 62, pl. 3. Let AT meet the tangent in H and join CF, then becauſe FC is equal to CA and the angle TCH to ACH, the triangles FCH, ACH are equal in every reſpect, therefore the angle AHC is equal to the angle FHC, and AF is perpendicular to the tangent CH.

SECOND SOLUTION, *by Mr.* JOHN WHITLEY.

Fig. 63, pl. 3. Let GH, a ſtraight line drawn through the focus F, of an ellipſe PHL, meet the directrix GT in G, the point in which the diameter PL meets it; the ſtraight line GH, ſhall be perpendicular to a tangent at either extremity of the diameter PL. Draw PS to the other focus S, and through F, the ſtraight line LFI. Then becauſe SC is equal to FC, and PC to LC, therefore PS is equal and parallel to FL; wherefore the angle IFP is equal to the angle SPF. Draw PO perpendicular to a tangent to the curve at P or L: then PO will biſect the angle SPF (cor. to prop. x. B. I. Emerſon's Conics). And (by prop. LXIII. ibid.) the ſtraight line GF alſo biſects the angle IFP; therefore the angle GFP is equal to the angle OPF; and conſequently GF is parallel to PO, or perpendicular to a tangent at either extremity of the diameter PL. And the ſame is manifeſtly true of any other conic ſection whatever.

XV. QUESTION 165, *by* AMICUS.

If one end of a given beam reſt againſt a ſmooth vertical plane, and the other end be ſuſtained by a weight faſtened to a ſtring, which paſſes over a pulley placed at a given point; it is required to determine the poſition of the beam when it reſts in equilibrio?

First Solution, *by Mr.* Lowry.

Let AB (fig. 64, pl. 3.) repreſent the beam, C its centre of gravity, EG the vertical plane, P the pulley given in poſition, W the weight faſtened to the end of the ſtring, and *w* the weight of the beam. Let the vertical line IC meet AP in I and join BI; then the beam is acted upon by three forces, *viz.* the weight W acting at A in the direction AI, the weight *w* acting at C in the direction IC, and the reaction of the plane againſt the end of the beam at B, acting perpendicularly to the plane; and as theſe forces keep the beam in equilibrio they neceſſarily tend to the ſame point I, or the reaction of the plane is exerted in the direction IB, and conſequently IB is perpendicular to EG.

Draw AD parallel to IB, meeting IC produced in D; then the forces acting at A, C, and B in the directions AI, IC, and IB are proportional to the lines AI, ID and AD; or if AI repreſent the weight W, then ID will repreſent the weight *w*, and AD will repreſent the preſſure at B.

Produce AP to meet the plane at E and draw PQ perpendicular to EG; then, ſince the pulley and vertical plane are given in poſition, the line PQ will be given in magnitude, and the point Q will be given in poſition; and the right-angled triangles QEP, AID being ſimilar, PE : EQ :: AI : ID :: W : *w*, or $PE = EQ \times \frac{W}{w}$; alſo $PE^2 - EQ^2 = PQ^2$, or $EQ^2 \times \frac{W^2}{w^2} - EQ^2 = PQ^2$; and therefore $EQ = PQ \times \frac{w}{\sqrt{(W^2 - w^2)}}$, which is a given quantity; therefore E is a given point, and the line EPA is given by poſition, or the angle PEQ (AID) is of a given magnitude.

The poſition of AB may now be determined either by conſtruction, or by calculation:

And firſt by conſtruction. Take a line *ab* = to AB the length of the beam, and let *ac* be = to AC and *cb* = to CB; on *ac* deſcribe a ſegment of a circle to contain the given angle AID or PEQ, and let it meet the ſemi-circle deſcribed on *cb* in *i*; then if *ai*, *bi*, *ci* be drawn, they will evidently be equal to the lines AI, BI, CI, reſpectively. On QP produced take QR = *bi*, and draw RI parallel to QG, meeting AP in I; take QB = RI and IA = *ai*, and the points A and B are determined. This conſtruction is too obvious to need any particular demonſtration in this place.

By calculation. Put $PQ = b$, $EQ = a = b \times \frac{w}{\sqrt{(W^2 - w^2)}}$, $AC = m$, $BC = n$, $AB = l$, and $QB = x$: Then by ſimilar triangles, $AB : BE :: AC : IC$,

$$\text{or } l : a + x :: m : \frac{m}{l} \times (a + x) = IC.$$

Alſo $EQ : PQ :: BE : IB$,

$$\text{or } a : b :: a + x : \frac{b}{a} \times (a + x) = IB.$$

And $IC^2 + IB^2 = CB^2$, or $\left(\frac{m^2}{l^2} + \frac{b^2}{a^2}\right)(a + x)^2 = n^2$;

$$\text{Therefore } (a + x)^2 = \frac{n^2}{\frac{m^2}{l^2} + \frac{b^2}{a^2}} = \frac{a^2 l^2 n^2}{m^2 a^2 + l^2 b^2},$$

$$\text{or } a + x = \frac{aln}{\sqrt{(m^2 a^2 + l^2 b^2}}, \text{ and } x = \frac{aln}{\sqrt{(m^2 a^2 + l^2 b^2}} - a =$$

$$w \left\{ \frac{ln}{\sqrt{\{l^2(W^2 - w^2) + m^2 w^2\}}} - \frac{b}{\sqrt{(W^2 - w^2)}} \right\}.$$

From this equation we may determine x when the beam and the weight w are given; or we may find what weight W is neceſſary to retain the beam in any aſſigned poſition.

In what has been done, the plane has been conſidered as perfectly ſmooth, but the problem may be reſolved with the ſame eaſe when the friction at any point of the plane is proportional to the preſſure of the beam at that point.

In this caſe, let AB (fig. 65, pl. 3.) repreſent the beam, and draw AE, IC and PQ as before. Let BH, drawn perpendicular to the plane, repreſent the preſſure at B, and take BL to BH in the proportion of the friction to the preſſure (as f to 1): Then BL will repreſent the friction, or a force equivalent to it, acting in the direction of the plane and oppoſing the deſcent of the end of the beam. Complete the parallelogram B H F L; then BF will be the reſultant of the forces BH, BL; and when the beam is in equilibrio the line BF muſt paſs through the point I. Draw AD parallel to ID meeting IC produced in D, and PK parallel to IB meeting the plane in K; Then $PQ : QK :: FL : BL :: 1 : f$, or $PK = f \times PQ$ is a given line, and the angle $PKE = IBL = CIB$ is a given angle. Again, the forces acting in the directions AI, IC and IB, are proportional to the lines AI, ID, and AD, or, becauſe of the ſimilar triangles PEK, AID, $PE : EK :: AI : ID :: W : w$, which is a given ratio; therefore the triangle PKE is given in ſpecies, and ſince PK is given, the line EK and the angle PEK (= AIC) are alſo given, and the line EA is given in poſition. Therefore if two circular ſegments be deſcribed on the lines *ac*, *cb*, to contain the given angles AIC, CIB and interſect at *i*, the lines AI, BI will be determined, and the remaining part of the conſtruction will be preciſely the ſame as before.

Calculation. Put PQ $= b$, EK $= a$, PK $= d$, and $x =$ KB.

Then, QK : PQ :: BL : LF :: $1 : f$, or QK $= \frac{b}{f}$ and PK $= \sqrt{\left(b^2 + \frac{b^2}{f^2}\right)} = \frac{b}{f}\sqrt{(f^2+1)} = d$. Alſo, the ſine of the angle PKQ $= \frac{f}{\sqrt{(f^2+1)}} = s$, and its coſine $= \frac{1}{\sqrt{(f^2+1)}} = c$. Again, $w : W :: a : \frac{W}{w}a =$ IE, and by trigonometry, PK $= d = ac + \frac{a}{w}\sqrt{(W^2 - s^2w^2)}$, or $a = \frac{dw}{cw + \sqrt{(W^2 - s^2w^2)}}$.

Now, by ſimilar triangles,

$$l : a+x :: m : \frac{m}{l} \times (a+x) = IC,$$

$$a : d :: a+x : \frac{d}{a} \times (a+x) = IB;$$

and by trigonometry, $IC^2 + IB^2 - 2c \times IC \times IB = CB^2 = n^2$, or $\left(\frac{m^2}{l^2} + \frac{d^2}{a^2}\right)(a+x)^2 - 2c \times \frac{md}{la}(a+x)^2 = n^2$;

therefore $(a+x)^2 = \frac{n^2l^2a^2}{a^2m^2 + d^2l^2 - 2cmdla}$,

and $x = \frac{nla}{\sqrt{(a^2m^2 + d^2l^2 - 2mcdla)}} - a$.

Let AS be drawn perpendicular to ID, then if ID repreſent the weight of the beam, AS will repreſent the preſſure againſt the plane, and SD the friction. And, by ſimilar triangles,

EK : PQ :: ID : AS $= \frac{b}{a}w = \frac{b}{d}\left\{cw + \sqrt{(W^2 - s^2w^2)}\right\}$ the preſſure againſt the plane, and SD $= \frac{b}{fd}\left\{cw + \sqrt{(W^2 - s^2w^2)}\right\}$ the friction.

The problem may be reſolved in the ſame manner when the plane makes any given angle with the horizon, only the calculation will be more complicated.

SECOND SOLUTION, *by the Rev. Mr.* TOPLIS.

Let AD (fig. 66, pl. 3.) repreſent the given beam, DI the vertical plane, B the pulley given in poſition; ſuppoſe W the weight which is connected with A by a ſtring paſſing over B. From the centre of gravity C of the beam, draw CE repreſenting the weight of the beam; decompoſe the force CE into the forces CF, FE, one perpendicular and the other parallel to the beam; from D draw DN perpendicular to the beam and equal to CF × AC ÷ AD; from N draw NK perpendicular to ND, and conſequently parallel to DA, and from D draw DK perpendicular to DI and meeting NK in K; then DK will repreſent the reaction of the plane upon the beam, DN its reaction in a direction perpendicular to the beam, and NK its reaction in a direction parallel to it. From A draw AG perpendicular to the beam and equal to CF × CD ÷ AD, from G draw GH parallel to the beam and equal NK + FE, join AH, then AH will repreſent the power and direction of the weight W. In this caſe the ſyſtem, ſuppoſing it in a plane perpendicular to the horizon and vertical plane, is evidently in equilibrio; for, from the property of the lever, the perpendicular forces counteract each other, and the parallel forces NK, FE are oppoſed by the force GH which is equal to their ſum.

To find the poſition of the beam by calculation, let us ſuppoſe CE $= a$, ſine of the angle CEF $= x$, AC $= b$, CD $= c$, AD $= e$.

By trigonometry,

$$\text{Rad. } (1) : \text{ſin. CEF } (x) :: \text{CE } (a) : \text{CF} = ax.$$

$$\text{Rad. } (1) : \text{coſ. CEF } (\sqrt{1-x^2}) :: \text{CE } (a) : \text{EF} = a\sqrt{(1-x^2)},$$

But AG × AD = CF × CD, therefore $\text{AG} = \frac{\text{CF} \times \text{CD}}{\text{AD}} = \frac{acx}{e}$; likewiſe ND × AD = AC × CF, therefore $\text{ND} = \frac{\text{AC} \times \text{CF}}{\text{AD}} = \frac{abx}{e}$. As MD is parallel to CE and CF to ND the angle NDM is equal to the angle ECF, conſequently the angle NDK is equal to the angle CEF, each of them being the complements of equal angles; therefore as the triangles CEF, NKD have two angles in each equal they are ſimilar. In the triangle NDK, we have ND $= \frac{abx}{e}$, ſin. NKD = ſin. CEF $= x$, and tang. NDK $= \frac{\sqrt{(1-x^2)}}{x}$, and, by trigonometry, rad. (1) : tan. KDN $\left(\frac{\sqrt{(1-x^2)}}{x}\right)$ :: ND $\left(\frac{abx}{e}\right)$: NK $= \frac{ab\sqrt{(1-x^2)}}{e}$.

In the triangle AGH, we have AG $= \frac{acx}{e}$, GH = NK + FE $= \frac{ae+ab}{e}\sqrt{(1-x^2)}$, AH $= \sqrt{(AG^2 + GH^2)} = \sqrt{\left\{\frac{(e+b)^2}{e^2} \times a^2 \times (1-x^2) + \frac{a^2c^2x^2}{e^2}\right\}} = W$. If we ſuppoſe $(e+b)^2 \times \frac{a^2}{e^2} = f$, $\frac{a^2c^2}{e^2} = g$, the expreſſion becomes $\sqrt{\{f-(f-g)x^2\}} = W$, or $x = \sqrt{\{(f-W) \div (f-g)\}}$. From which equation if W is given, x or the ſine of CDI may be found; or if the ſine of CDI, the beam's inclination, is given, the weight W may be found.

Let DK be produced to meet BW in P, and draw BM parallel to PD, or perpendicular to the plane; then becauſe the pulley is given in poſition, the line BM or its equal PD is given, and the angle DSP (equal to the angle CEF) being found, the lines SP, SD may be found, and then AS becomes known.

Alſo, rad. : ſin. AHG = HAD :: AH : AG :: W : $\frac{acx}{e}$, or ſin. HAD $= \frac{acx}{eW} = \frac{ac}{eW} \times \sqrt{\frac{f-W}{f-g}}$, and therefore the angle HAD is known, and in the triangle ABS the angles and the ſide AS are known to find BS : then BS − PS = BP = MD the diſtance of the end of the beam from the point M.

XVI. QUESTION 166, *by* X. Y.

Suppoſe two weights connected by a ſtring, which paſſes over a pulley, and that one of the weights is placed upon an inclined plane, while the other hangs in the air by the connecting ſtring. It is required to determine the *locus* of the centre of gravity of the weights, ſuppoſing the relative poſition of the pulley and plane, the length of the ſtring, and the weights to be all given?

FIRST SOLUTION, *by Mr.* WALLACE.

Let *a* and *b* be the weights, (fig. 67, pl. 3.) and APB the ſtring connecting them, which paſſes over the pulley P. And inſtead of confining the ſolution of the problem to the caſe in which one of the weights B is placed upon an inclined plane, let us ſuppoſe it placed upon a given line HK of any kind whatever.

Let C be the centre of gravity of the weights. Draw CD parallel to the vertical line PA, meeting PB in D: Then, from ſimilar triangles, AP + PB is to CD + DB as AB to CB, that is, from the nature of the centre of gravity, as *a* + *b* to *a*; hence the ratio of AP + PB to CD + DB is given; and becauſe AP + PB is given, CD + DB is alſo given. In the vertical line PA take PE = CD + DB, and then E will be a given point. Draw EF parallel to PB, meeting DC in F, and becauſe DF = PE = DC + DB, therefore CF = BD; now EF = PD, and PD : DB :: AC : CB :: *b* : *a*, therefore EF : FC :: *b* : *a*; hence EF has to FC a given ratio. Draw BG parallel to PA, and PG parallel to EC, meeting BG in G; then becauſe DF is equal to DC + DB, ſo that the point F is below the point C, therefore the point B is alſo below the point G; and becauſe the triangles PBG, EFC are ſimilar, PB : BG :: EF : FC; but the ratio of EF to FC has been ſhewn to be given, therefore the ratio of PB to BG is alſo given. Now P being a given point, and HBK a line given in poſition, it is manifeſt that the *locus* of the point G will be a line MGN of a determinate kind, which will depend upon the nature of the given line HBK. Again, becauſe PG is to EC as PB to EF or PD, that is, as AB to AC or as *a* + *b* to *b*, therefore PG has to EC a given ratio; now PG is alſo parallel to EC, and E is a given point, therefore, whatever be the nature of the line MGN which is the *locus* of the point G, the point C will always be in a line *m*C*n* ſimilar to it, and ſimilarly ſituated; ſo that the nature of the line MGN being once known, that of *m*C*n* the locus of the centre of gravity of the weights will alſo become known.

The ſolution of the problem thus generalized, which reſults from the analyſis is therefore ſhortly this: Draw PB to any point B in the line HK; draw BG parallel to PA and in a contrary direction, and take BG ſuch, that PB may have to BG the given ratio of *b* to *a*, then determine the nature of MN the *locus* of the point G, and the nature of *mn* the line which is the locus of C, the centre of gravity of the weights will alſo be determined, for it is ſimilar to MN and ſimilarly ſituated. And if PE be taken a fourth proportional to *a* + *b*, *a*, and AP + PB, then the given points P and E will be ſimilarly ſituated in reſpect of the ſimilar lines MN, *mn*; that is, if from P and E any two parallel ſtraight lines be drawn meeting them in G and C, PG will have to EC a given ratio.

Let us apply this general ſolution to ſome particular caſes, and firſt, let us ſuppoſe, as in the queſtion, that the weight *b* is placed at any point B (fig. 68, pl. 3.) upon HBK a ſtraight line given in poſition. Draw PI perpendicular to HK, and IV parallel to PA, and take IV of ſuch a magnitude that PI : IV :: $b : a$ :: PB : BG; then I and V will evidently be given points, and V a point in MN the locus of G; and becauſe IV : BG :: PI : PB, therefore $IV^2 : BG^2 :: PI^2 : PI^2 + IB^2$. Now PI and IV are both given in magnitude, and BG makes a given angle with HK, therefore the point G is manifeſtly in a hyperbola of which I is the centre, IV a ſemi-tranſverſe diameter, and HK its conjugate; and it alſo appears that the ſemi-conjugate diameter is equal to PI. Hence, *mn* the *locus* of C is a hyperbola, and from what has been ſhewn every thing relative to it may be readily determined.

Next let us ſuppoſe that the weight *b* is placed at B (fig. 69, pl. 3.), upon HK an arch of a conic ſection, and that the pulley P is at the focus. It is eaſy to ſee that we may aſſume ſuch a ratio for the given ratio of *b* to *a*, or of PB to BG, that MN (the *locus* of the point G) ſhall be the directrix of the ſection, and in this caſe the locus of the centre of gravity of the weights is a ſtraight line. To determine the ratio of *a* to *b*, in this caſe, let γ denote the angle which AP makes with the directrix, and ſuppoſe the ſemi-tranſverſe axis to be to the excentricity as 1 to *e*, then, drawing BL perpendicular to the directrix, we have BG $= \frac{BL}{\text{ſin. } \gamma}$, and ſince by hypotheſis $b : a :: PB : BG$, and from the nature of a conic ſection PB : BL :: $e : 1$, therefore $a : b :: 1 : e \text{ ſin. } \gamma$. If *a* have to *b* any other ratio than that of 1 to e ſin. γ the locus of G and conſequently of C the centre of gravity is no longer a ſtraight line but a conic ſection, as it is very eaſy to ſhew.

It is probable that the queſtion, which we have juſt now reſolved, was firſt ſuggeſted by this other problem. The ſame things being ſuppoſed as in the queſtion, with this difference that one of the weights, inſtead of being placed upon an inclined plane, reſts upon a curve of any kind. It is required to determine the poſition of the weights when they reſt in equilibrio. To this problem the preceding analyſis points out an eaſy ſolution. For draw GL (fig. 70. pl. 3.) perpendicular to PA. Then, becauſe the weights will be in equilibrio when C their centre of gravity is as low as it can deſcend, and becauſe when C is the loweſt poſſible it appears from what has been ſhewn that G will alſo be the loweſt poſſible; it is evident the weights will be in equilibrio when PL is a maximum or minimum. Draw BQ perpendicular to PA; Put $PQ = x$, $QB = y$; then, becauſe $b : a :: PB : BG$, we have BG or QL equal to $\frac{a}{b} \sqrt{(x^2 + y^2)}$, and PL $= \frac{a}{b} \sqrt{(x^2 + y^2)} - x$, a maximum or minimum; hence, taking the fluxions, &c. we find

$$\frac{a}{b}\left\{x + \frac{\dot{y}y}{\dot{x}}\right\} = \sqrt{(x^2 + y^2)}.$$

Draw BR a normal to the curve HK, then it is manifeſt that $QR = \frac{\dot{y}y}{\dot{x}}$ and $PR = x + \frac{\dot{y}y}{\dot{x}}$, now $PB = \sqrt{(x^2 + y^2)}$, therefore the weights will be in equilibrio, whatever be the nature of the curve HK, when $\frac{a}{b} \times PR = PB$, that is, when

$$PB : PR :: a : b.$$

This concluſion agrees, as it ought to do, with what may be otherwiſe found by the reſolution of forces.

SECOND SOLUTION, *by Mr.* LOWRY.

Fig. 71, pl. 3. Let AB be the inclined plane, C the poſition of the pulley, G the centre of gravity of the given weights P, Q, and l the length of the ſtring. Draw GE parallel to the vertical line CP and CR perpendicular to the plane AB meeting EF, a parallel to AB, in F; then becauſe the relative poſition of the pulley and plane is given, the line CR is given; and by the property of the centre of gravity, as the weight P : the weight Q :: GQ : GP, that is as EQ : EC :: RF : FC; therefore RF and FC are given lines. Again, as the ſum of the weights P, Q is to the weight P :: PQ : GQ :: $PC + CQ = l$: $GE + EQ = \frac{P}{P+Q} \times l$ a given line. On a line drawn through F, parallel to CP, take $FL = \frac{P}{P+Q} \times l$, then L is a given point. Draw GS parallel and EK perpendicular to the plane AB, and take LH = to EK or FR; then becauſe FL is = GE + EQ, and FS = EG, LS is = to EQ; and by ſimilar triangles,

EF

$$EF = SG : KQ :: FC : EK = LH,$$

or $SG^2 : KQ^2 = QE^2 - KE^2 = LS^2 - LH^2 :: FC^2 : LH^2$, which is the property of a hyperbola paſſing through H and having its centre at L, the lines HL and LI (a parallel to SG and = to FC) being two ſemi-conjugate diameters.

When the weights are equal, and the plane is horizontal, the hyperbola is equilateral. And when the pulley is placed at A the locus is then a ſtraight line.

THIRD SOLUTION, *by Mr.* CUNLIFFE.

Let P be the pulley (fig. 72, pl. 3), AB the inclined plane, m and n the weights placed in any aſſigned poſition, G their centre of gravity, and l the length of the ſtring. Draw GO parallel to AB and put PA $= a$, the ſine and coſine of the angle RAB $= s$ and c, AO $= x$, OG $= y$, and the ſum of the weights m and $n = w$. Then, by the property of the centre of gravity,

$$m : n :: Gn : Gm :: AC : Cm = \frac{n}{m}x,$$

herefore $Pm = a + x + \frac{n}{m}x = a + \frac{w}{m}x.$

Again, $w : n :: Gn + Gm : Gm :: An : OG$,

or $An = \frac{w}{n}y$; and by trigonometry,

$$Pn = \sqrt{(AP^2 + An^2 + 2cAP \cdot An)} = \sqrt{(a^2 + \frac{w^2}{n^2}y^2 + \frac{2caw}{n}y)};$$

therefore $Pn + Pm = \sqrt{(a^2 + \frac{w^2}{n^2}y^2 + \frac{2caw}{n}y)} + a + \frac{w}{m}x = l$,

$$\text{or } a^2 + \frac{w^2}{n^2}y^2 + \frac{2caw}{n}y = (l - a - \frac{w}{m}x)^2;$$

and by reducing, we have

$$y^2 + \frac{2nca}{w}y - \frac{n^2}{m^2}x^2 + \frac{2n^2}{mw}(l - a)x - \frac{n^2}{w^2}(l^2 - 2al) = 0,$$

which is an equation to a hyperbola.

To determine the poſition of the curve, aſſume any point C for its centre, and draw CD parallel to mP meeting GO, BA produced in I and D, and the curve in L. Put AD $= d$, DC $= b$, the ſemi-diameter CL $= t$, and the parameter to that diameter $= p$; then IG $= d + y$ and CI $= b - x$, and by the property of the hyperbola,

$$IG^2 : CI^2 - CL^2 :: p : 2t,$$

or $(d+y)^2 : (c-x)^2 - t^2 :: p : 2t$; which analogy being reduced to an equation, gives

$$2^2 + 2dy - \frac{p}{2t}x + \frac{p}{t}cx + d^2 + \frac{p}{2t}(t^2 - c^2) = 0.$$

Comparing this equation with the given one, we have $2d = \frac{ynca}{w}$, or $d = \frac{nca}{w}$; $\frac{p}{2t} = \frac{n^2}{m^2}$; $\frac{p}{t}c = \frac{2n^2}{m^2}c = \frac{2n^2}{mw}(l-a)$, or $c = \frac{m}{w}(l-a)$, and $d^2 + \frac{p}{2t}(t^2 - c^2) = -\frac{n^2}{w^2}(l^2 - 2al)$, or

$$t^2 = c^2 - \frac{m^2}{w^2}(l^2 - 2al) - \frac{m^2}{n^2}d^2 = \frac{m^2}{w^2}a^2(1 - c^2),$$

(putting for c and d their values found above), and conſequently $t = \frac{m}{w}as$. Alſo, ſince $\frac{p}{2t} = \frac{n^2}{m^2}$, we have $p = \frac{2n^2}{m^2} \times t = \frac{2n^2}{mw} \times sa$.

Now, let AD be taken = to $\frac{n}{w}ca$, and on a line drawn parallel to Pm, take OC $= \frac{m}{w}(l-a)$, and CL $= \frac{m}{w} \times sa$; then C is the centre of the hyperbola, CL a ſemi-diameter, and $\frac{2n^2}{mw} \times sa$ the parameter of that diameter.

When a is $= 0$, the equation becomes,

$$y^2 - \frac{n^2}{m^2}x^2 + \frac{2n^2}{mw}lx - \frac{n^2}{w^2}l^2 = 0,$$

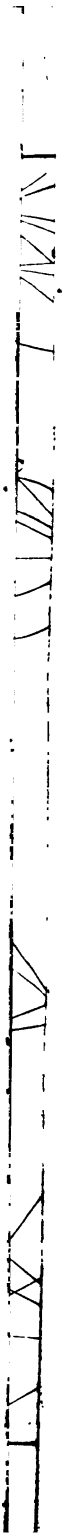
$$\text{or } y^2 = \frac{n^2}{m^2}x^2 - \frac{2n^2}{mw}lx + \frac{n^2}{w^2}l^2,$$

and $y = \pm\left(\frac{n}{m}x - \frac{n}{w}l\right)$ which is an equation to a ſtraight line.

When y is $= 0$, x is $= \frac{m}{w}l$, and when x is $= 0$, y is $= \frac{n}{w}l$; therefore $\frac{m}{w}l$ and $\frac{n}{w}l$ are the diſtances from the point A, where the locus meets the line Am and the plane AB.

The Rev. J. Toplis *alſo ſent a ſolution to this queſtion.*

XVII.



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XVII. QUESTION 167, *by* LIMENUS.

When any number of circles paſs through the ſame two points, tangents drawn from any point in either of the two external circumferences to the remaining circles ſhall be to one another in a given ratio?

SOLUTION, *by* ———.

Let there be any number of circles PQA, PQB, PQC, PQD, &c, (fig. 73, pl. 4.) paſſing through the two points P, Q: then, if from any point A in the exterior ſegment PQA, tangents be drawn to the other circles they will have a given ratio. Let a, b, c, d; &c. be the centres of the given circles PQA, PQB, PQC, PQD, &c. which will evidently be in the ſame ſtraight line; through the points A, Q, draw a line to meet the circles in B, C, D, &c. and draw PA, PB, PC, PD, &c. and Pa, Pb, Pc, Pd, &c. then the angles PAQ, PBQ, PCQ, PDQ, &c. being reſpectively equal to the angles Paq, Pbq, Pcq, Pdq, &c. the triangles PAB, PAC, PAD, &c. are reſpectively ſimilar to the triangles Pab, Pac, Pad, &c. and conſequently the line AD is divided ſimilarly to the line ad, or the lines AB, AC, AD, &c. have the ſame ratio as the lines ab, ac, ad, &c. But the ſquares of the tangents drawn from A to the circles, are equal to the rectangles AQ·AB, AQ·AC, AQ·AD, &c. or proportional to the lines AB, AC, AD, &c. or to the lines ab, ac, ad, &c. as is proved above; and therefore the tangents are in the ſubduplicate ratio of the given diſtances ab, ac, ad, &c.

XVIII. QUESTION 168, *by* LAPUTIENSIS.

Suppoſe two bodies, whoſe weights are known to be connected by a flexible line or ſtring, paſſing through a ſmall hole in a perfectly ſmooth horizontal plane, and let the body which is above the plane be projected thereon, with a given velocity at a given diſtance from the hole, in a direction perpendicular to the ſtring which is kept tight by the other body hanging freely; it is required to determine the velocity, time, and angle formed by the ſtring with the firſt poſition, when the projected body has attained any other poſſible aſſigned diſtance from the hole through which the ſtring paſſes?

SOLUTION,

SOLUTION, *by Mr.* CUNLIFFE.

Let A and B (fig. 74, pl. 4.) repreſent the two bodies, the body B being that which is projected upon the horizontal plane, and the body A that which hangs freely below the plane by means of the connecting ſtring BCA paſſing through the hole at C.

Let the curve BB′*b* repreſent the track of the body B, B′ and *b* being two points therein indefinitely near to each other: draw CB′, C*b* and with the centre C and radius CB′ deſcribe the ſmall arch B′*d* cutting C*b* in *d*.

Put $CB = a$, $CB' = s$, $db = \dot{s}$, $z =$ circular arc (radius $CB = a$) intercepted between CB, CB′; $v =$ the velocity at B′ in the direction B′C or CB′; $t =$ time of deſcribing BB′, and and $g = 32\frac{1}{6}$; alſo let f denote the projectile velocity at B perpendicular to CB. By the property of tangents, we ſhall have $B'b : B'd :: B'C : B'C \times Bd \div B'b =$ a perpendicular from C upon a tangent to the curve at B′, and $bd : B'b :: v : B'b \times v \div bd =$ the velocity at B′ in the direction of the curve.

Now, it is well known, that the velocities of a revolving body in different points of its orbit, are inverſely as perpendiculars from the centre of force upon tangents to the orbit at thoſe points; wherefore $f : \frac{B'b \times v}{bd} :: \frac{B'C \times B'd}{B'b} : BC$, whence $\frac{B'd \times v}{bd} = \frac{BC \times f}{B'C} = \frac{af}{s}$. It is manifeſt that $\frac{B'd \times v}{bd}$ expreſſes the velocity of B′ in the direction B′*d*, or the circulatory velocity at B′: and hence the centrifugal force at B′ is expreſſed by $\frac{a^2f^2}{s^3}$, and therefore the centrifugal force of the body B at that point will be $\frac{a^2f^2}{s^3} \times B$; from whence, deducting A*g*, the gravity of A, leaves $\frac{a^2f^2}{s^3} \times B - Ag$, for the motive force urging both bodies in the direction CB′ or B′C; and conſequently $\left(\frac{a^2f^2}{s^3} \times B - Ag\right) \div (A + B)$ will expreſs the accelerative force in the ſame direction. Then, by a known theorem,

$$\left\{\left(\frac{a^2f^2}{s^3} \times B - Ag\right) \div (A + B)\right\} \times \dot{s} = v\dot{v},$$

and taking the fluents,

$$-\left(\frac{a^2f^2}{2s^2}\times \mathrm{B} + \mathrm{A}gs\right) \div (\mathrm{A}+\mathrm{B}) = \frac{v^2}{2};$$

but when $s = a$, $v = 0$, therefore the correct equation of the fluents is,

$$\left(\frac{f^2\times \mathrm{B}}{2} + \mathrm{A}ga - \frac{a^2f^2}{2s^2}\times \mathrm{B} - \mathrm{A}gs\right) \div (\mathrm{A}+\mathrm{B}) = \frac{v^2}{2}$$

$$\text{or, } v^2 = \frac{a-s}{s^2}\cdot\left\{\frac{2\mathrm{A}gs^2}{\mathrm{A}+\mathrm{B}} - \frac{\mathrm{B}f^2(a+s)}{\mathrm{A}+\mathrm{B}}\right\} = \frac{a-s}{s^2}\cdot\left\{2mgs^2 - f^2(1-m)(a+s)\right\}$$

by putting $m = \frac{\mathrm{A}}{\mathrm{A}+\mathrm{B}}$, or $\frac{v^2}{2mg} = \frac{a-s}{s^2}\times\left\{s^2 - \frac{f^2(1-m)(a+s)}{2mg}\right\}$.

It may be remarked that when $g = 0$, the preceding expreſſion for the velocity becomes $f^2 \times \left(\frac{s^2-a^2}{s^2}\right) \times \frac{\mathrm{B}}{\mathrm{A}+\mathrm{B}}$ or $v^2 \times \frac{\mathrm{A}+\mathrm{B}}{\mathrm{B}} = f^2 \times \left(\frac{s^2-a^2}{s^2}\right)$ which is the very expreſſion for the velocity obtained in the ſolution to queſtion 105, vol. 1, and is therefore a confirmation of the truth of the concluſions, both in that and the preſent problem.'

We ſhall, moreover, have $\dot{t} = \frac{s\dot{s}\sqrt{2mg}}{\sqrt{(a-s)\cdot\left\{s^2 - \frac{f^2(1-m)(a+s)}{2mg}\right\}}}$ the fluent of which may be had by infinite ſeries, and correcting the ſame, by making the reſulting expreſſion to vaniſh when $s = a$.

The angle BCB' may be determined from the equation $\frac{\mathrm{B}'d\times v}{bd} = \frac{af}{s}$ before deduced,

For, from thence $\mathrm{B}'d = \frac{af\dot{s}}{sv}$: alſo $a : \dot{z} :: s : \frac{s\dot{z}}{a} = \mathrm{B}'d$, therefore $\frac{s\dot{z}}{a} = \frac{af\dot{s}}{sv}$ and $\dot{z} = \frac{a^2f\dot{s}}{s^2v} =$

$$\frac{a^2f\dot{s}\sqrt{2mg}}{\sqrt{(a-s)}\times\left\{s^2 - \frac{f^2(1-m)(a+s)}{2mg}\right\}}$$ the fluent of which

may alſo be had by infinite ſeries, which is to be corrected by making it to vaniſh when $s = a$, as obſerved in the preceding caſe.

A ſolution to queſtion 105, vol. 1, might have been conciſely effected as follows: We have determined that the motive centrifugal force at B′, urging both bodies in the direction CB′, is expreſſed by $\frac{a^2 f^2}{s^3} \times \mathrm{B}$, and therefore the accelerative force is $\frac{a^2 f^2}{s^3} \times \frac{\mathrm{B}}{\mathrm{A} + \mathrm{B}}$; and hence, by a known theorem, $\frac{a^2 f^2 \dot{s}}{s^2} \times \frac{\mathrm{B}}{\mathrm{A}+\mathrm{B}} = v\dot{v}$, the fluent of which is $-\frac{a^2 f^2}{2s^2} \times \frac{\mathrm{B}}{\mathrm{A} + \mathrm{B}} = \frac{v^2}{2}$, or $-\frac{a^2 f^2}{s^2} \times \frac{\mathrm{B}}{\mathrm{A}+\mathrm{B}} = v^2$; but when $s = a$, $v = 0$, therefore the correct equation of the fluents is

$$\left(f^2 - \frac{a^2 f^2}{s^2}\right) \times \frac{\mathrm{B}}{\mathrm{A}+\mathrm{B}} = f^2 \times \left(\frac{s^2 - a^2}{s^2}\right) \times \frac{\mathrm{B}}{\mathrm{A}+\mathrm{B}} = v^2.$$

And thus the Queſtion was anſwered by Mr. Bazley.

SECOND SOLUTION, *by Mr.* LOWRY,

Let M be the weight (or maſs) of the projected body, and m that of the other body: alſo let C (fig. 75, pl. 4.) be the hole through which the ſtring paſſes, A the poſition of the projected body at the commencement of the motion, and M its poſition at the end of any indefinite time t, AMb being a portion of the curve deſcribed by the body on the horizontal plane. Put $a =$ CA, the initial diſtance of the body from the hole; b the velocity of projection, or the ſpace deſcribed by the projected body in a ſecond of time; $r =$ CM the radius vector of the revolving body, and $g = 32\frac{1}{6}$ feet the meaſure of the accelerative force of gravity. Then the projected body is acted upon by two forces, that is, by a motive force $= mg$, acting in the direction of the ſtring MC and urging the body towards the centre C; and by a motive force Mf (f being the centrifugal force at M) acting alſo in the direction of the ſtring but urging the body to recede from the centre; therefore the whole motive force acting on the maſs M in the direction of the ſtring is $=$ M$f - mg$. To determine the force f, let the point b be another poſition of the body indefinitely near to M; join Cb, and with the diſtance CM, and centre C, deſcribe the arc Ma. Draw CB perpendicular to the tangent at M, let u be the velocity of the body at M in the curve Mb, and w its velocity in the arc Ma. Then the triangles baM, ABM may be conſidered as ſimilar, and $w : u :: a\mathrm{M} : b\mathrm{M} :: \mathrm{CB} : \mathrm{CM} = r$,

or $w = \frac{u \times \mathrm{CB}}{r}$.

But the velocities of the revolving body at the points A, M are inverſely as the perpendiculars on the tangents at thoſe points, or as CB to CA; therefore $u \times \mathrm{CB} = b \times \mathrm{AC} = ba$, and

conſequently $w = \frac{ba}{r}$. Therefore f is $= \frac{w^2}{r} =$ to $\frac{b^2a^2}{r^3}$;

and the whole motive force = to $\frac{\mathrm{M}b^2a^2}{r^3} - mg$. Therefore the

whole accelerative force is $= \frac{1}{\mathrm{M}+m}\left(\frac{\mathrm{M}b^2a^2}{r^3} - mg\right)$,

and we have $\ddot{r} = \frac{\dot{t}^2}{\mathrm{M}+m} \times \left(\frac{\mathrm{M}b^2a^2}{r^3} - mg\right)$.

Multiply this equation by $2\dot{r}$,

then $\dot{r}\ddot{r} = \dot{t}^2 \times \left(\frac{2\mathrm{M}b^2a^2\dot{r}}{r^3} - 2mg\dot{r}\right)$;

and taking the fluents $\dot{r}^2 = \frac{\dot{t}^2}{\mathrm{M}+m} \times (\mathrm{C} - \frac{\mathrm{M}b^2a^2}{r^2} - 2mgr)$.

Now at the commencement of the motion, or when $t = 0$, the quantity included between the parentheſis is alſo = to 0, and r being then = to a, it is obvious that C is $= \mathrm{M}b^2 + 2mga$;

therefore $\dot{r}^2 = \frac{\dot{t}^2}{\mathrm{M}+m} \times (\mathrm{M}b^2 + 2mga - \frac{\mathrm{M}b^2a^2}{r^2} - 2mgr)$

and $\dot{t} = \frac{(\mathrm{M}+m)^{\frac{1}{2}}r\dot{r}}{\sqrt{\{(\mathrm{M}b^2 + 2mga)r^2 - 2gmr^3 - \mathrm{M}b^2a^2\}}}$;

where t may be aſſigned by means of elliptic arcs.

Again $\frac{\dot{r}}{\dot{t}} = \frac{1}{(\mathrm{M}+m)^{\frac{1}{2}}r} \times \sqrt{\{(\mathrm{M}b^2+2mga)r^2 - 2gmr^3 - \mathrm{M}b^2a^2\}}$

is the velocity of the body in the direction of the ſtring.

Let q be the angle that the ſtring makes with the line from which the motion commences (reckoned on the arc of the circle whoſe radius is 1), then $r : 1 :: $ the arc $Ma = w\dot{t} : \dot{q}$, or

$$\dot{q} = \frac{ka}{r^2}\,\dot{t} = \frac{(M + m)^{\frac{1}{2}}\,ba\dot{r}}{\sqrt{\left\{(Mb^2 + 2mga)\,r^2 - 2gmr^3 - Mb^2a^2\right\}}}.$$

The fluent, however, cannot be generally aſſigned without having recourſe to an infinite ſeries, or to ſome curve of a higher order than the conic ſections.

XIX. QUESTION 169, *by* Hermodorus.

Given the vaniſhing line, its centre and diſtance, to find the perſpective repreſentation of a circle from the given repreſentation of one radius, or of an inſcribed triangle?

Solution, *by* 1550.

Let PQ (fig. 76, pl. 4.) be the given vaniſhing line, its centre O and its diſtance NO perpendicular to PQ, then from the other data it appears by Brook Taylor's Perſpective, 8vo. 1715, Ex. 3 and 4, and Hamilton's Perſpective, p. 80, that the repreſentation of that diameter which tends to the centre O of the vaniſhing line may be found; let this be MK paſſing through I the centre of the circle, and let H be one of the points of the circle given in the queſtion, and join MH, and KH meeting PQ in F and G, and draw HE parallel to PQ meeting OM in E.

Then ſince KM repreſents a diameter, KHM repreſents a right angle, therefore GNF is a right angle; alſo by ſimilar triangles OM : ME :: OF : EH, and OK : KE :: OG : EH; therefore MO × OK : ME × EK :: ON² (= FO × OG) : EH², which is the property of a conic ſection, and from which, as MO × OK and ON² are given, it may be conſtructed by known methods.

Cor. By combing this ſolution with that in the 8th. Example of Brook Taylor's Perſpective, 8vo. 1715, it is perfectly eaſy to find what is the repreſentation of a ſphere in any given poſition whatever and to conſtruct the conic ſection.

N. B. There are three editions of Dr. Brook Taylor's Perſpective, *viz.* in 1715, 1719, 1749, all in 8vo. and all ſcarce books; the two laſt are alike even to the preſs errors, but the firſt contains many things which the ſecond does not, and the ſecond that the firſt does not.

The reader may likewiſe conſult Hamilton's Perſpective, where he will find good conſtructions of theſe and ſeveral ſimilar problems, as well as many other curious matters relating to Perſpective: that book has no fault but its great prolixity, but it will amply reward the learned draftsman's patience by the complete knowledge of the theory and practice of Perſpective, which he will learn from it.

XX. QUESTION 170, *by* HERMODORUS.

Given in poſition a circle and two points; if from the two points be inflected two ſtraight lines to any point of the circle, and from the other two points in which theſe lines meet the circle, there be inflected two other ſtraight lines to a point in the circle, ſo that one of them may be parallel to a ſtraight line given by poſition, then the other ſtraight line will either tend to a given point, or make a given angle with a ſtraight line tending to a given point?

SOLUTION, *by* ———.

Let A, B (fig. 77, pl. 4.) be two given points, and CDE a circle given by poſition; if from the points A, B two ſtraight lines AC, BC be inflected to any point C of the circle, meeting it in D and E; and from the points D, E two other ſtraight lines DH, EH be inflected to any point H in the circle, ſo that one of them, EH, may be parallel to a ſtraight line given by poſition; then the ſtraight line DH, either tends to a given point, or makes a given angle with a ſtraight line tending to a given point.

This is evident from the 57th. and 62d. Propoſitions of Dr. Simſon's Treatiſe de Poriſmatibus, or which is the ſame, prob. iii. art. ix. vol. i, part ii. For, let the rectangle KAB be made equal to the given rectangle DAC, and then K will be a given point. Draw the diameter KIS, and divide it in F, ſo that IF : FS :: IK : KS, then F will alſo be a given point. Let DF be drawn to meet the circle in G and H. Then, by the 57th propoſition, if the lines AC, BC be inflected to any point C of the circle, meeting it in D and E, the line joining the points D, E either paſſes through the point F, or makes a given angle at D with a ſtraight line tending to that point, that is the angle EDG, or its equal the angle EHG, is given; and therefore ſince IE is parallel to a ſtraight line given by poſition, GH is alſo parallel to a ſtraight line given by poſition: therefore by the 62d. propoſition, ſince a line DG is drawn through the given point F meeting the

circle

be adopted to denote the aggregate ſum of all the poſſible combinations, made by multiplying together m quantities out of theſe n quantities, *viz.* $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots\ldots \frac{1}{n}$, ſo that, according to this notation, $\left(\frac{n}{1}\right)$ will denote the ſum of theſe n fractions; and $\frac{n}{2}$ will denote the ſum of all the products that ariſe by combining them by twos; and $\left(\frac{n}{3}\right)$ will denote the ſum of all the products that ariſe by combining them by threes; and ſo on. Then, by what is commonly taught concerning the coefficients of binomial products in algebra, the *Terminus Generalis*, when developed and arranged according to the powers of x, will become

$$\pm \frac{a^{n+1}}{n+1} . x \mp \left(\frac{n}{1}\right) . \frac{a^{n+1}}{n+1} . x^2 \pm \left(\frac{n}{2}\right) . \frac{a^{n+1}}{n+1} . x^3 \mp \left(\frac{n}{3}\right) . \frac{a^{n+1}}{n+1} . x^4$$

$\pm$ &c. the upper or lower ſign taking place according as n is even or odd.

If, now, each term in the ſeries that is equal to $(1 + a)^x$ be expanded as we have juſt been explaining, and if all the terms be arranged according to the powers of x, we ſhall obtain,

$$\begin{aligned}(1+a)^x = 1 + x \times & \left\{ a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \frac{a^5}{5} - \&c. \right. \\ + x^2 \times & \left\{ \left(\tfrac{1}{1}\right) . \frac{a^2}{2} - \left(\tfrac{2}{1}\right) . \frac{a^3}{3} + \left(\tfrac{3}{1}\right) . \frac{a^4}{4} - \&c. \right. \\ + x^3 \times & \left\{ \left(\tfrac{2}{2}\right) . \frac{a^3}{3} - \left(\tfrac{3}{2}\right) . \frac{a^4}{4} + \left(\tfrac{4}{2}\right) . \frac{a^5}{5} - \&c, \right. \\ + x^4 \times & \left\{ \left(\tfrac{3}{3}\right) . \frac{a^4}{4} - \left(\tfrac{4}{3}\right) . \frac{a^5}{5} + \left(\tfrac{5}{3}\right) . \frac{a^6}{6} - \&c, \right. \\ & \&c. \ \&c.\end{aligned}$$

The *Terminus Generalis* of this ſeries being

$$\mp x^n \times \left\{ \left(\frac{n-1}{n-1}\right) . \frac{a^n}{n} - \left(\frac{n}{n-1}\right) . \frac{a^{n+1}}{n+1} + \left(\frac{n+1}{n-1}\right) . \frac{a^{n+2}}{n+2} - \&c. \right\},$$

Let τ denote the hyp. log. of $1 + a$, then $(1 + a)^x$ (conſidered as a function of x) being expanded in the uſual manner, we get

$$(1+a)^x = 1 + \frac{\tau}{1} . x + \frac{\tau^2}{1.2} . x^2 + \frac{\tau^3}{1.2.3} . x^3 + \frac{\tau^4}{1.2.3.4} . x^4 + \&c.$$

+ &c. the *Terminus Generalis* being $\frac{\tau^n}{1 . 2 . 3 \ldots . n} \times x^n$.

But the coefficients of the ſame powers of x in the two developements muſt be equal to one another; therefore

$$\frac{\tau^n}{1.2.3 \ldots . n} = \left(\frac{n-1}{n-1}\right) . \frac{a^n}{n} - \left(\frac{n}{n-1}\right) . \frac{a^{n+1}}{n+1} + \left(\frac{n+1}{n-1}\right) . \frac{a^{n+2}}{n+2} - \&c.$$

and conſequently τ^n, or the n^{th} power of the logarithm of $(1 + a)$, =

$$1.2.3 \ldots . n \times \left\{ \left(\frac{n-1}{n-1}\right) . \frac{a^n}{n} - \left(\frac{n}{n-1}\right) . \frac{a^{n+1}}{n+1} + \left(\frac{n+1}{n-1}\right) . \frac{a^{n+2}}{n+2} - \&c. \right.$$

If we make $n = 2$, then becauſe the expreſſions $(\frac{1}{1})$, $(\frac{2}{1})$, $(\frac{3}{1})$, &c. denote the ſums of one, two, and three, &c. of the quantities 1, $\frac{1}{2}$, $\frac{1}{3}$, &c. we have the ſquare of the logarithm of $(1 + a) =$

$$1.2 \times \left\{ \frac{a^2}{2} - (1 + \tfrac{1}{2}) . \frac{a^3}{3} + (1 + \tfrac{1}{2} + \tfrac{1}{3}) . \frac{a^4}{4} - \&c. \right\}.$$

And if we make $n = 3$, then the cube of the logarithm of $(1 + a) =$

$$1.2.3 \times \left\{ \frac{1}{1.2} . \frac{a^3}{3} - (\tfrac{3}{2}) . \frac{a^4}{4} + (\tfrac{4}{2}) . \frac{a^5}{5} - (\tfrac{5}{2}) . \frac{a^6}{6} + \&c. \right\}$$

where the expreſſions $(\frac{3}{2})$, $(\frac{4}{2})$, $(\frac{5}{2})$, &c. denote the ſum of the products of the couplets of three, and four, and five, &c. of the quantities 1, $\frac{4}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c.

And if we make $n = 4$, then the fourth power of the logarithm of $(1 + a) =$

$$1.2.3.4 \times \left\{ \frac{1}{1.2.3} . \frac{a^4}{4} - (\tfrac{4}{3}) . \frac{a^5}{5} + (\tfrac{5}{3}) . \frac{a^6}{6} - (\tfrac{6}{3}) . \frac{a^7}{7} + \&c. \right\}$$

where the expreſſions $(\frac{4}{3})$, $(\frac{5}{3})$, $(\frac{6}{3})$, &c. denote the ſum of the products of all the triplets of four, and five, and ſix, &c. of the quantities 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c.

The method we have here uſed, will apply only when the index of the required power is a whole poſitive number. If the index be negative, or fractional, we may then have recourſe to the ordinary method of aſſuming a ſeries with indeterminate coefficients.

The Rev. Mr. Toplis *likewiſe anſwered it.*

XXII. QUESTION 172, *by* SENEX.

It is required to find the nature or equation of a curve ſuch, that any right line RQP being drawn through a given point P, to cut the curve in two points R and Q, and tangents from theſe points produced to interſect each other in I, the angle QIR may always be of the ſame conſtant magnitude, that is always equal to a given angle?

FIRST SOLUTION, *by Mr.* IVORY.

Fig. 78, pl. 4. Let RT and QS be perpendicular to a fixed axis drawn through P; put VP, the diſtance of the point P from a given point V in the axis, $= b$, VT $= x$, VS $= x'$, TR $= y$, SQ $= y'$. Let the tangents RI and QI meet the axis in H and K: and let $a =$ tangent of half the angle QIR ($=$ ſum of the angles RHT and QKS), and $z =$ tangent of half the difference of the angles RHT and QKS: then tangent of RHT $= \frac{a+z}{1-az}$, and tangent of QKS $= \frac{a-z}{1+az}$. But $\frac{dy}{dx} =$ tangent of RHT, and $\frac{dy'}{dx'} =$ tangent of QKS. Therefore $\frac{dx}{dy} = \frac{1-az}{a+z}$ and $\frac{dx'}{dy'} = \frac{1+az}{a-z}$. Again, becauſe the triangles TPR and QPS are equiangular, therefore $\frac{b-x}{y} = \frac{x'-b}{y'}$: and by taking fluxions,

$$\frac{dy}{y} \cdot \left\{ -\frac{dx}{dy} - \frac{b-x}{y} \right\} = \frac{dy'}{y'} \cdot \left\{ \frac{dx'}{dy'} - \frac{x'-b}{y'} \right\}.$$

Hence, obſerving that $\frac{b-x}{y} = \frac{x'-b}{y'}$, we get

$$\left(\frac{dy}{y} - \frac{dy'}{y'}\right) \cdot \frac{b-x}{y} = -\frac{dy}{y} \cdot \frac{dx}{dy} - \frac{dy'}{y'} \cdot \frac{dx'}{dy'},$$

$$\left(\frac{dy}{y} - \frac{dy'}{y'}\right) \cdot \frac{x'-b}{y'} = -\frac{dy}{y} \cdot \frac{dx}{dy} - \frac{dy'}{y'} \cdot \frac{dx'}{dy'}$$

or, which is the ſame thing,

$$2\left(\frac{dy}{y}-\frac{dy'}{y'}\right).\frac{b-x}{y}=-\left(\frac{dy}{y}+\frac{dy'}{y'}\right).\left(\frac{dx}{dy}+\frac{dx'}{dy'}\right)-\left(\frac{dy}{y}-\frac{dy'}{y'}\right).\left(\frac{dx}{dy}-\frac{dx'}{dy'}\right),$$

$$2\left(\frac{dy}{y}-\frac{dy'}{y'}\right).\frac{x'-b}{y'}=-\left(\frac{dy}{y}+\frac{dy'}{y'}\right).\left(\frac{dx}{dy}+\frac{dx'}{dy'}\right)-\left(\frac{dy}{y}-\frac{dy'}{y'}\right).\left(\frac{dx}{dy}-\frac{dx'}{dy'}\right),$$

But $\frac{dx}{dy}+\frac{dx'}{dy'}=\frac{2a(1+z^2)}{a^2-z^2}$, and $\frac{dx}{dy}-\frac{dx'}{dy'}=\frac{-2z(1+a^2)}{a^2-z^2}$; therefore, if we put $\frac{dy}{y}+\frac{dy'}{y'}=\left(\frac{dy}{y}-\frac{dy'}{y'}\right)\times u$, we ſhall get

$$\frac{b-x}{y}=-u\times\frac{a(1+z^2)}{a^2-z^2}+\frac{z(1+a^2)}{a^2-z^2};$$

$$\frac{x'-b}{y'}=-u\times\frac{a(1+z^2)}{a^2-z^2}+\frac{z(1+a^2)}{a^2-z^2};$$

whence, we derive,

$$x=b+uy.\frac{a(1+z^2)}{a^2-z^2}-y.\frac{z(1+a^2)}{a^2-z^2};$$

$$x'=b-uy'.\frac{a(1+z^2)}{a^2-z^2}+y'.\frac{z(1+a^2)}{a^2-z^2}.$$

And if we take the fluxions of theſe equations, and ſubſtitute $dy\times\frac{1-az}{a+z}$ for dx and $dy'\times\frac{1+az}{a-z}$ for dx'; then after having properly ordered the terms, we ſhall get

$$\frac{dy}{y}=\frac{du}{1-u}+\frac{u}{1-u}.\frac{(1+a^2)2zdz}{(1+z^2)(a^2-z^2)}-\frac{a^2+1}{a}.\frac{(a^2+z^2)dz}{(1+u)(1+z^2)(a^2-z^2)}$$

$$\frac{dy'}{y'}=\frac{-du}{1+u}-\frac{u}{1+u}.\frac{(1+a^2)2zdz}{(1+z^2)(a^2-z^2)}+\frac{a^2+1}{a}.\frac{(a^2+z^2)dz}{(1+u)(1+z^2)(a^2-z^2)}.$$

Theſe equations contain all the conditions of the queſtion; for they include the equation $\frac{dy}{y}+\frac{dy'}{y'}=u\times\left(\frac{dy}{y}-\frac{dy'}{y'}\right)$, and, by this means the function u is left indefinite or arbitrary. Becauſe x', y' and x, y are co-ordinates of the ſame curve, therefore the two former muſt ſatisfy the equation of the curve equally as the two latter do; which requires that x' and y' be reſpectively the ſame functions of $-z$, that x and y are of $+z$. For, when this is the caſe, it is obvious that, by exterminating z from the expreſſions x' and y', there will be obtained the ſame equation between x' and y', as is obtained by exterminating z from the expreſſions x and y. This conſideration reſtricts the nature of the functions of z that muſt be ſubſtituted for u: for only ſuch functions of z muſt be ſubſtituted for u as will make the expreſſions for y and y' like functions of $+z$ and $-z$. Without entering into further detail it will be ſufficient to remark that the required conditions will be fulfilled; that is, the expreſſions for y and y' will be like functions of $+z$ and $-z$; if there be ſubſtituted for u, functions of this form, *viz.* $Az + Bz^3 + Cz^5 + \&c.$; or ſuch functions of z as, when expanded into ſeries, contain only the odd powers of z. Having thus aſcertained the proper functions of z to be ſubſtituted for u, we have only to conſider the expreſſions of y and x, *viz.*

$$\frac{dy}{y} = \frac{du}{1-u} + \frac{u}{1-u} \cdot \frac{(1+a^2)\,2zdz}{(1+z^2)(a^2-z^2)} - \frac{1+a^2}{a} \cdot \frac{(a^2+z^2)\,dz}{(1-u)(1+z^2)(a^2-z^2)},$$

$$x = b + uy \times \frac{a\,(1+z^2)}{a^2-z^2} - y \times \frac{(1+a^2)\,z}{a^2-z^2}.$$

In looking for the ſimpleſt curves that will ſatisfy the queſtion, we are naturally led to examine the caſe, when $u = 0$. It is manifeſt from what has been obſerved, that this ſuppoſition is permitted. Now, when $u = 0$, then $\frac{dy}{y} = -\frac{1+a^2}{a} \cdot \frac{(a^2+z^2)\,dz}{(1+z^2)(a^2-z^2)}$: And, by reſolving into parts, to prepare for integrating, $\frac{dy}{y} = -\frac{a^2-1}{a} \cdot \frac{dz}{1+z^2} - \frac{2adz}{a^2-z^2}$: and by putting $m = \frac{a^2-1}{a} = a - \frac{1}{a}$, and then integrating, $\log. y = \log. c - m \times \varphi + \log. \frac{a-z}{a+z}$, where φ = arch whoſe tangent is z: therefore if e be the number whoſe hyp. log. is 1, then $y = c \cdot e^{-m\varphi} \times \frac{a-z}{a+z}$. But, when $u = 0$, then $x = b - y \times \frac{(1+a^2)z}{a^2-z^2} = b - c\,(1+a^2) \cdot e^{-m\varphi} \times \frac{z}{(a+z)^2} = b - \frac{c\,(1+a^2)}{4a} \cdot e^{-m\varphi} \times \frac{4az}{(a+z)^2}$. Let τ denote the limit of φ, or the arch whoſe tangent is a, and let c be determined, ſo that x and y may vaniſh together; then $b = \frac{c\,(1+a^2)}{4a} \cdot e^{-m\tau}$; and, by ſubſtitution, $x = \frac{c\,(1+a)}{4a} \cdot e^{-m\tau} \times \left\{1 - e^{m\,(\tau-\varphi)} \times \frac{4az}{(a+z)^2}\right\}$; and, by changing b to denote $\frac{c\,(1+a^2)}{4a} \cdot e^{-m\tau}$, the expreſſions for the co-ordinates of the curve become, $x = b - be^{m\,(\tau - \varphi)} \times \frac{4az}{(a+z)^2}$, and $y = \frac{4ab}{1+a^2} \cdot e^{m\,(\tau - \varphi)} \times \frac{a-z}{a+z}$; where a is the tangent of half the angle τ made by the tangents of the curve; $m = a - \frac{1}{a}$; tangent $\tau = a$; tangent $\varphi = z$, the limits of φ being $\pm\,\tau$, and of z, $\pm\,a$.

If $a = 1$ (in which caſe the angle made by the tangents is a right-angle) then $m = 0$, and the expreſſions of the co-ordinates become, $x = b - b \cdot \frac{4z}{(1+z)^2}$ and $y = 2b \cdot \frac{1-z}{1+z}$: therefore, in this caſe the curve is algebraical; and if z be exterminated, we get $y^2 = 4bx$, the equation of a conic parabola having the parameter of the axis $= 4b$. The reſult of the analyſis, in this caſe, when expreſſed in words, is this; "that tangents of a parabola drawn from the extremities of any line paſſing through the focus, interſect one another at right angles."

The ſuppoſition, next to that of $u = 0$, in point of ſimplicity, is to make $u = pz$. This ſubſtitution being made in the expreſſions of $\frac{dy}{y}$ and x, we get

$$\frac{dy}{y} = \frac{p \cdot dz}{1 - pz} + \frac{1+a^2}{a} \cdot \frac{(2paz^2 - a^2 - z^2) \cdot dz}{(1-pz)(1+z^2)(a^2 - z^2)}.$$

$$x = b + y \times \frac{(pa - 1 - a^2 + pa \times z^2)\,z}{a^2 - z^2}$$

It would be eaſy to examine if theſe new formulas comprehend any algebraic curves that deſerve to be remarked on account of their ſimplicity. But, waving this diſcuſſion, we ſhall be content to notice the caſe, when $pa = 1$, or $p = \frac{1}{a}$; the formulas then become,

$$\frac{dy}{y} = \frac{dz}{a - z} - \frac{(1+a^2)\,dz}{(a-z)(1+z^2)} = \frac{-adz}{1+z^2} - \frac{zdz}{1+z^2}$$

$$x = b - y \times z.$$

And by integrating, log. y = log. $b + a\int\frac{-dz}{1+z^2}$ − log. $\sqrt{(1+z^2)}$: and if tangent $\varphi = \frac{1}{z}$; then $y = b \times e^{a\varphi} \times$ ſin. φ.

Whence $x = b - b \times e^{a\varphi}$ coſ. φ. And if ϱ denote PR, the diſtance of the point of the curve, of which x and y are the co-ordinates from P, then $\varrho = \left\{ y^2 + (b - x)^2 \right\} = be^{a\varphi}$; the equation of the common logarithmic ſpiral.

SECOND SOLUTION, *by Mr.* WALLACE.

Let QBR (fig. 79, pl. 4.) be a curve having the property required, and let its nature be expreſſed by an equation expreſſing the relation between PQ, a ſtraight line drawn from P to any point of the curve, and ſome function of the angle contained by PQ, and PA, a line given by poſition. Put φ for the angle APQ, and as the angle PQI muſt be expreſſible in ſome way or other by the angle APQ, let the angle PQI be denoted by $f\varphi$, where by $f\varphi$ is meant a function of φ, of a determinate form, but as yet unknown.

Now, if we ſuppoſe the angle PQI to revolve about the point P, (the line PR remaining at reſt) till the point Q arrive at R, then the tangent QI will coincide with RK, that is, with the tangent IR produced; and, as by ſuch revolution, the angle APQ will have changed its value from φ to $\varphi + \pi$, (here π is put for two right-angles) while the angle PQI has changed its value ſo as to become PRK, it follows that whatever function the angle PQI is of φ, the ſame function will the angle PRK be of $\varphi + \pi$; therefore the angle PRK will be $f\,(\varphi + \pi)$. Put α for the angle I, which by the queſtion is to be a conſtant quantity, then, becauſe the angle I is the difference of the angles PRK, PQI, we have

$$f\,(\varphi + \pi) - f\varphi = \alpha,$$

an equation reſolvable by the theory of *finite differences*, and which, employing the notation of that method, may be otherwiſe expreſſed thus;

$$\Delta\,(f\varphi) = \alpha.$$

Hence, taking the integral, and obſerving that $\Delta\varphi = \pi$, we have

$$f\varphi = \frac{\alpha}{\pi}\varphi + \mathrm{F}\ (\text{ſin. } 2\varphi,\ \text{coſ. } 2\varphi);$$

where by F (ſin. 2φ, coſ. 2φ) is meant any function whatever of the ſine and coſine of 2φ; let this arbitrary function be denoted by ψ, then, putting $n\pi$ for α, ſo that $\frac{\alpha}{\pi} = n$, we have

$$\angle\ \mathrm{PQI} = f\varphi = n\varphi + \psi.$$

Draw Pq indefinitely near to PQ and qs perpendcular to PQ; let PQ be denoted by r; then, putting dr and $d\varphi$ for the differentials or fluxions of r and φ, we have Q$s = \pm\, dr$, and $sq = rd\varphi$; and ſince tan. ∠ PQI $= \frac{sq}{s\mathrm{Q}}$, therefore tan. $(n\varphi + \psi) = \frac{rd\varphi}{\pm\, dr}$, and hence

$$\frac{dr}{r} = \frac{\pm\, d\varphi}{\text{tan.}\ (n\varphi + \psi)}.$$

Now, before we can integrate this equation we muſt give ſome determinate form to the arbitrary function ψ, and this being done, the reſult will have this form,

$$\frac{dr}{r} = \Phi d\varphi,$$

where by Φ is meant ſome known function of φ, and hence, by integrating, we ſhall have an equation expreſſing the nature of the curve ſought. And as there is no limit to the number of different forms which may be given to the arbitrary function ψ, and conſequently to the number of correſponding forms of the function Φ, it is evident that there are innumerable curves having the property required by the queſtion.

In giving different forms to the function ψ, it may happen that ſome of them are ſuch as to render the equation $\frac{dr}{r} = \Phi d\varphi$ capable of having its integral expreſſed in finite terms. This, however, will not be the caſe in general, and perhaps it may not be eaſy to aſſign ſuch forms to the function ψ, as ſhall produce a finite integral.

If we ſuppoſe that $\psi = \beta$, a given angle, then the fluxional equation becomes,

$$\frac{dr}{r} = \frac{\pm\, d\varphi}{\text{tan.}\ (n\varphi + \beta)} = \frac{\pm 1}{n} \cdot \frac{\text{coſ.}\ (n\varphi + \beta)\ dn\varphi}{\text{ſin.}\ (n\varphi + \beta)};$$

and hence, by integrating, and putting c for a conſtant quantity, we get

$$r = c\ \text{ſin.}^{\pm\frac{1}{n}}\ (n\varphi + \beta),$$

an equation which expreſſes the nature of an indefinite number of curves having the property required by the queſtion.

Let us ſuppoſe α to be a right-angle, and aſſume $\beta = 0$, then, becauſe in this caſe $n = \frac{\alpha}{\pi} = \frac{1}{2}$, we have

$$r = c\ \text{ſin.}^{\pm 2}\ \tfrac{1}{2}\varphi.$$

If we give the negative ſign to the exponent, we have

$$r = \frac{2c}{2\ \text{ſin.}^2\ \frac{1}{2}\varphi} = \frac{2c}{1 - \text{coſ.}\ \varphi},$$

an equation which is known to belong to a parabola (fig. 79), having P for its focus, and PA for its axis.

If, again, we ſuppoſe the exponent to be poſitive, we have

$$r = c\ \text{ſin.}^2\tfrac{1}{2}\varphi = \frac{c}{2}(1 - \text{coſ.}\ \varphi).$$

This equation belongs to an epcycloid PQBR (fig. 80, pl. 4.) generated by a given point Q, in the circumference of a circle DQ, which revolves on the outſide of another circle PD of equal magnitude, the given point Q, in the revolving circle being ſuppoſed to depart from P, which is alſo the point of contact of the two circles at the beginning of the motion, and the diameter of each being equal to the conſtant quantity c. For ſuppoſe that Q is a point in the curve thus generated; draw CDE through the centres of the circles and their point of contact, and draw QE, PF perpendicular to CD. Then, becauſe *arch* PD $=$ *arch* QD, the ſtraight line PQ is parallel to CE, and PQ $=$ FE $=$ 2FD. But becauſe CD $=$ CP, and CF $=$ CP $\times$ coſ. PCF $=$ PC $\times -$ coſ. APQ, therefore PQ $=$ CP $\times$ (1 $-$ coſ. APQ), that is, $r = \frac{c}{2}(1 - \text{coſ.}\ \varphi)$.

If we reſume the equation

$$\frac{dr}{r} = \frac{\pm\, d\varphi}{\tan.\ (n\varphi + \psi)};$$

and ſuppoſe $\alpha = 0$, that is, if we ſuppoſe the tangents at the extremities of the line QR to be parallel, then, as in this caſe, $n = \frac{a}{\pi} = 0$, we have, (aſſuming $\psi =$ a conſtant angle, and putting m for its tangent)

$$\frac{dr}{r} = \pm \frac{1}{m}\, d\varphi,$$

hence, by integrating, and putting c for a conſtant quantity, we get

$$\log.\ r = c\varphi,$$

an equation to the logarithmic ſpiral.

XXIII. QUESTION 173, *by Mr.* JOHN FLETCHER.

If the magnitude of a circle is given; it is required to find the poſition of a right line RS, and that of a point P in another line given in poſition, from whence, if PAB be drawn to cut the circle in A and B, and perpendiculars AR and BS are drawn to RS, their rectangle will be always a given magnitude?

SOLUTION, *by Mr.* LOWRY.

This problem is eaſily reſolved by means of the following lemma, which is the ſame as prop. xviii. of Dr. Stewart's Propoſitiones Geometricæ.

Fig. 81, 82, pl. 4. If two points C, P, be taken in the diameter EF of a circle, ſo that the ſquare of PC may be equal to the rectangle ECF, and a line SCK be drawn perpendicular to PF; then if any ſtraight line be drawn from P to meet the circle in A and B, and perpendiculars AR, BS, be drawn to the line SK, the rectangle of thoſe perpendiculars will be equal to the ſquare of PC.

Let PB meet SK in L, and draw LE to meet the circle in Q; then $ALB = ELQ = CL^2 + ECF = CL^2 + CP^2 = PL^2$; and by ſimilar triangles,

$$AR : AL :: PC : PL,$$
$$BS : LB :: PC : PL;$$
$$\text{therefore } AR \cdot BS : ALB = PL^2 :: PC^2 : PL^2,$$
$$\text{and } AR \cdot BS = PC^2.$$

This being premiſed, let O be the centre of the circle, produce the diameter FE till the rectangle ECF be equal to the given ſpace, or to the ſquare of the line n; and then the diſtance CO will be given. On FC produced take CP = to n, and then the diſtance CP will be given: with this diſtance and centre O, deſcribe an arc to cut the given line in P, and then SCK, drawn through C, perpendicular to PO, is the line required.

For, by the lemma, the rectangle $AR \cdot BS$ is always equal to PC^2 or to the given ſpace n^2.

Mr. Cunliffe *alſo anſwered this queſtion.*

XXIV. QUESTION 174, *by* AMICUS.

If a given cone, ſuſpended by its vertex, be made to revolve with its axis in a vertical plane; it is required to determine the force exerted on the centre of ſuſpenſion at any given poſition of the revolving body?

Solution, *by Mr.* Lowry.

Inſtead of confining the ſolution to the particular caſe of the cone, as propoſed in the queſtion, I ſhall reſolve the problem generally, when the body is any homogeneous ſolid formed by the revolution of a plane figure about an axis.

Let ASB (fig. 83, pl. 4.) be a ſection of the body made by the vertical plane in which the axis SC oſcillates or revolves; alſo, let G be the centre of gravity, O the centre of oſcillation, and S the centre of ſuſpenſion of the body. Let p be the meaſure of the angle (reckoned on the periphery of a circle whoſe radius is 1) that the axis makes with the horizontal line SF, when the body is in any aſſigned poſition; and put SG $= a$, SO $= b$, $g = 32\frac{1}{6}$ feet, the meaſure of the accelerative force of gravity, and w the weight of the body.

Then the force of gravity (or weight w) may be reſolved into two forces, the one equal to ſin. $p \times w$ acting in the direction of the axis SC, and the other equal to coſ. $p \times w$ acting perpendicularly to SC. Now the force ſin. $p \times w$ acting in the direction SC, can have no effect on the motion of the body, but is wholly exerted on the centre of ſuſpenſion, and, when reduced to the horizontal direction SF, it is equal to ſin. $p \times$ coſ. $p \times w$.

Again, the force coſ. $p \times w$, acting perpendicularly to SC, is that which accelerates the motion of the body about the centre, ſuppoſing the whole maſs of the body to be concentrated at O, but the abſolute velocities produced at G and O being directly as the diſtances a, b, from the centre of motion, the velocity at G is only the $\frac{a}{b}$ part of the velocity at O, and therefore the force which is employed at G, in accelerating the motion of the body is only the $\frac{a}{b}$ part of the force coſ. $p \times w$, therefore the remaining part of this force, or $\frac{b-a}{b}$ coſ. $p \times w$, is exerted on the axis of ſuſpenſion in a direction perpendicular to SC. This part of the force, when reduced to the horizontal direction FS, is $=$ to $\frac{b-a}{b}$ ſin. p coſ. $p \times w$, and as it acts in a contrary direction to the other horizontal force ſin. p coſ. $p \times w$, we have $\left(1 - \frac{b-a}{b}\right)$ ſin. p coſ. $p \times w$, or $\frac{a}{b}$ ſin. p coſ. $p \times w$, for the whole effect of gravity on the centre of ſuſpenſion in the horizontal direction SF.

But this is not the whole of the force that acts upon the centre of ſuſpenſion, for it is alſo affected by the centrifugal force of the body, and to compute the effect of this force, let f be the centrifugal force of a particle at O, and v its velocity; then the forces of an equal particle, at the diſtances d, d', d'', &c. from the axis of motion, are equal to $\frac{f}{b} \times d, \frac{f}{b} \times d', \frac{f}{b} \times d''$ &c. therefore the centrifugal force of the whole body is equal to $\frac{f}{b}$ multiplied by the ſum of the products of each particle by its diſtance from the axis of motion, or (by the property of the centre of gravity) = to $\frac{fa}{b}$ multiplied by the maſs of the body.

Now, by the nature of central forces, the force f is equal to $\frac{v^2}{b}$, and to eliminate v, let a circle be deſcribed about S with the diſtance SO, meeting the vertical line that paſſes through S in D and E, and draw OI parallel to FS. Then, if the body revolves about the centre S, let HE be taken equal to the height due to the velocity of the body at the loweſt point E; but if the body only oſcillates through a part of the circle, as through the arc EK, then let KH be drawn parallel to FS meeting the vertical line in H; in either caſe HI will be the height due to the velocity of the body at O, or v^2 will be equal to $2g \times$ HI. But SI is $= b \times$ ſin. p, and putting SH $= c$, we have HI $= b \times$ ſin. $p \pm c$, and $v^2 = 2g$ (b ſin. $p \pm c$) the upper or lower ſign taking place as the point H is above or below the centre; therefore $\frac{f}{b}$ is $= 2g \left(\frac{b \text{ ſin. } p \pm c}{b^2}\right)$. But the weight of the body is equal to g multiplied by the maſs, therefore $\frac{fa}{b}$ multiplied by the maſs is equal to $w \times \frac{2a}{b^2}$ (b ſin. $p \pm c$), the whole centrifugal force of the body in the direction SO; which being reduced to the horizontal direction SF is $= w \times \frac{2a \text{ coſ. } p}{b^2}$ (b ſin. $p \pm c$).

Let this force be added to the force of gravity found above, and we have

$$w \times \frac{a}{b} (\text{ſin. } p \text{ coſ. } p) + w \times \frac{2a \text{ coſ. } p}{b^2} (b \text{ ſin. } p \pm c), \text{ or}$$

$w \times \frac{a}{b} \left\{ (3\ \text{ſin.}\ p\ \text{coſ.}\ p) \pm \frac{2c}{b}\ \text{coſ.}\ p \right\}$ for the whole force required, when eſtimated in a horizontal direction.

To find the value of the angle p when this force is the greateſt, let the fluxion of the variable part of the above expreſſion be taken, and we have

$$3\ \text{ſin.}\ \dot{p}\ \text{coſ.}\ p + 3\ \text{coſ.}\ \dot{p}\ \text{ſin.}\ p \pm \frac{2c}{b}\ \text{coſ.}\ \dot{p} = 0\,;$$

ſubſtitute for ſin. $\dot{p}$ and coſ. $\dot{p}$, their values $\dot{p}$ coſ. p and $-\dot{p}$. ſin. p, and divide by $3\dot{p}$,

$$\text{then coſ.}^2\ p - \text{ſin.}^2\ p \mp \frac{2c}{3b}\ \text{ſin.}\ p = 0,$$

$$\text{or ſin.}^2\ p \pm \frac{c}{3b}\ \text{ſin.}\ p = \tfrac{1}{2},$$

$$\text{and ſin.}\ p = \frac{\sqrt{(18b + c^2)} \pm c}{6b}.$$

The force in the direction SF, ariſing from gravity being reduced to the vertical direction SE, is evidently equal to $w \times \frac{a}{b}$ ſin.^{2} p; and the centrifugal force, when reduced to the ſame direction, is equal to $w \times \frac{a}{b}$ ſin. p (b ſin. $p \pm c$); and therefore the whole force acting on the centre of ſuſpenſion in the vertical direction SE, is

$$= w \times \frac{a}{b}\ (3\ \text{ſin.}^2\ p \pm \frac{2c}{b}\ \text{ſin.}\ p).$$

When p is a quadrant, or the axis of the body coincides with the line SE, the force is $= w \times \frac{a}{b} \left(3 \pm \frac{2c}{b}\right)$; and when the points G and O coincide (as in the caſe of a ball ſuſpended by a ſtring) and the body oſcillates through a quadrant, c is then $= 0$, and the whole force $= 3w$, as determined in the ſolution to the 106th queſtion.

In the caſe of the cone, if d be the altitude, and r the radius of the baſe; then a is $= \frac{3}{4}d$, and $b = \frac{4d^2 + r^2}{5d}$, as is well known; and the expreſſion for the horizontal force, is

$w \times \frac{15d^2}{4(4d^2+r^2)}\left(3\ \text{ſin.}\ p\ \text{coſ.}\ p \pm \frac{10cd}{4d^2+r^2}\ \text{coſ.}\ p\right)$, and for the vertical force $w \times \frac{15d^2}{4(4d^2+r^2)}\left(3\ \text{ſin.}^2\ p \pm \frac{10cd}{4d^2+r^2}\ \text{ſin.}\ p\right)$.

When the cone oſcillates only through the arc EK, then, theſe expreſſions may be a little ſimplified, for ſince SH is the coſine of the arc EK, if q be put for the meaſure of the angle ESK (to the radius 1), then SH or c is $= b\ \text{coſ.}\ q$, and $\frac{2c}{b} = 2\ \text{coſ.}\ q$, and the above expreſſions for the forces become

$$w \times \frac{15d^2}{4(4d^2+r^2)}(3\ \text{ſin.}\ p\ \text{coſ.}\ p \pm 2\ \text{coſ.}\ q\ \text{coſ.}\ p),$$

and $$w \times \frac{15d^2}{4d^2+r^2}(3\ \text{ſin.}^2 p \pm 2\ \text{coſ.}\ q\ \text{ſin.}\ p).$$

XXV. QUESTION 175, *by Mr.* WALLACE, *R. M. College.*

Suppoſe a quadrilateral figure to be inſcribed in another quadrilateral figure, agreeable to the conditions required in queſtion 120 of the New Series of the Repoſitory, then the oppoſite ſides of the inſcribed figure when produced, will interſect each other in the diagonals produced of the circumſcribed figure. Required the demonſtration?

SOLUTION, *by Mr.* WILLIAM SIMPSON, *Bolton.*

Let *abcd* (fig. 84, pl. 4.) be a quadrilateral, inſcribed in ABCD agreeably to the conditions of the queſtion, and let the oppoſite ſides *ac*, *bd* meet in *g*; join B*g*, C*g* and draw the diagonal BC. Then it appears from the ſolution to queſtion 120 that *a*B biſects the angle *bag*, that *b*B biſects the exterior angle *abg*, that *d*C biſects the exterior angle *cd*M, and that C*c* biſects the exterior angle *dc*G; therefore B*g* biſects the angle *agb*, and C*g* biſects the angle *cgd*, as is pretty generally known (ſee Emerſon's Geometry 35, ii.). Therefore B*g* and C*g* biſect the ſame angle and conſequently coincide, and CB*g* is a ſtraight line. It is proved in the ſame manner that the other ſides and diagonal when produced meet in a point.

SECOND SOLUTION, *by Mr.* J. JOHNSON.

Let *abcd* (fig. 85, pl. 4.) be a quadrilateral inſcribed in another quadrilateral, ABCD, agreeably to the conditions required in queſtion 120. Then, if the oppoſite ſides *ab*, *cd*, when produced, interſect in the point H, it is to be proved that the diagonal AC, when produced, will alſo paſs through that point.

From A demit the perpendiculars AQ, AP, AR; then it appears from the ſolution to queſtion 120, that A*d* biſects the angle P*d*Q, and that A*a* biſects the angle R*a*Q; therefore the right-angled triangles AP*d*, AQ*d* are equiangular, and the ſide A*d* is common to each triangle, conſequently they are equal in all reſpects, and AP = AQ. For a like reaſon the triangles AR*a*, AQ*a* are equal in all reſpects, or AR = AQ; therefore AP is = AR.

Again if the perpendiculars CE, CF, CG, be drawn from C, it is proved in the ſame way as above that CG is = CF; therefore CG : CF :: AP : AR; and conſequently the points H, A, C are in a ſtraight line.

It is proved in the ſame manner that the diagonal BD will paſs through *h*, the point where the oppoſite ſides *bc*, *ad* interſect when produced.

The demonſtrations received from Meſſrs. Swale *and* Whitley, *are little different from thoſe above.*

XXVI. QUESTION 176, *by* MARLOVIENSIS.

It is well known that the ſum of the ſeries $x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \frac{x^5}{5^2} -$ &c. *ad infinitum*, is expreſſed by the fluent of $(x^{-1}\dot{x} \times \text{fl.} \frac{\dot{x}}{1+x})$, or area of a tranſcendent hyperbola. Under what circumſtances can the ſame ſum be exhibited by the multiples or powers of the arc of a circle, and how is this to be inveſtigated?

SOLUTION, *by Mr.* IVORY.

The inveſtigation of the ſums of the powers of the reciprocals of the natural numbers ſeems firſt to have engaged the attention of the two brothers, James and John Bernoulli. In a ſhort treatiſe on the ſums of infinite ſeries, written by James Bernoulli, (“ Tractatus de ſeriebus infinitis, &c.” publiſhed with the “ Ars Conjectandi” of the ſame author in one volume) we are informed (prop. 14) that John Bernoulli, firſt of all demonſtrated that the ſum of the infinite ſeries $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+$ &c. is greater than any finite number. In the ſame treatiſe, the ſum of the infinite ſeries,

$$\frac{1}{1\cdot3}+\frac{1}{2\cdot4}+\frac{1}{3\cdot5}+\frac{1}{4\cdot6}+\&c.=\frac{1}{2^2-1}+\frac{1}{3^2-1}+\frac{1}{4^2-1}$$

+ &c. is ſhewn to be equal to ¾: and from hence it follows that the ſum of the infinite ſeries $\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+$ &c. is a finite quantity leſs than ¾. But the ſummation of the ſeries $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+$ &c. was a problem for the ſolution of which the artifices then known were inſufficient: and James Bernoulli, ſpeaking of this ſeries, uſes the following words, “ difficilior eſt, quam quis expectaverit, ſummæ inveſtigatio; quam tamen finitam eſſe, ex altera, qua manifeſto minor eſt, colligimus: ſi quis inveniat, nobisque communicet, quod induſtriam eluſit hactenus, magnas de nobis gratias feret.” The further diſcoveries of the Bernoullis, on this ſubject, are confined to the demonſtration of the following properties; *viz.*

$$\left(1+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m}+\&c.\right)\times\frac{2^m-1}{2^m}=1+\frac{1}{3^m}+\frac{1}{5^m}+\frac{1}{7^m}+\&c.$$

$$\left(1+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m}+\&c.\right)\times\frac{2^{m-1}-1}{2^{m-1}}=1-\frac{1}{2^m}+\frac{1}{3^m}-\frac{1}{4^m}+\&c.$$

The ſummation of the ſeries $1+\frac{1}{2^2}+\frac{1}{3^2}+$ &c. was accompliſhed by Euler; but the diſcovery was too late to gratify the curioſity of James Bernoulli, who was dead. The method of Euler extends to the general ſeries $1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}$ + &c. where the index is any even number: and it is derived from the expreſſion obtained by reſolving the ſeries for the ſine of an arch into its binomial factors. Let z be any arch, and π the ſemi-circumference to the radius 1; then ſin. z

$$=z-\frac{z^3}{2\cdot3}+\frac{z^5}{2\cdot3\cdot4\cdot5}-\&c.=z\left(1-\frac{z^2}{\pi^2}\right)\left(1-\frac{z^2}{2^2\pi^2}\right)\left(1-\frac{z^2}{3^2\pi^2}\right)\&c.$$

and, by taking the fluxions of the logarithms of both ſides of this equation and dividing by dz,

$\frac{\text{coſ.}\, z}{\text{ſin.}\, z} = \frac{1}{z} - \frac{2z}{\pi^2 - z^2} - \frac{2z}{2^2\pi^2 - z^2} - \frac{2z}{3^2\pi^2 - z^2}$ - &c. let $\frac{\text{coſ.}\, z}{\text{ſin.}\, z} = \text{cot.}\, z = \frac{1}{z} - A^{(1)} z - A^{(3)} z^3 - A^{(5)} z^5$ - &c. the general term of the developement being $- A^{(2n-1)} z^{2n-1}$; and let the ſeveral fractions in the value of $\frac{\text{coſ.}\, z}{\text{ſin.}\, z}$ be developed into ſeries: then by equating the coefficients of the ſame powers of z, on different ſides of the equation, we obtain,

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c. = \frac{A^{(1)}}{2} . \pi^2 = \frac{\pi^2}{6}$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \&c. = \frac{A^{(3)}}{2} . \pi^4 = \frac{\pi^4}{90}$$

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \&c. = \frac{A^{(5)}}{2} . \pi^6 = \frac{\pi^6}{945}$$

and, in general, $1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \&c. = \frac{A^{(2n-1)}}{2} \times \pi^{2n}$.

From the value of the ſeries $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} +$ &c. we immediately derive,

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \&c. = \frac{\pi^2}{12}$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c. = \frac{\pi^2}{8}.$$

Theſe ſummations, firſt obtained by Euler, have been inveſtigated by other mathematicians in different ways; and, particularly, by Mr. Landen in various parts of his writings.

If we put $x = 1$, in the ſeries propoſed in the queſtion, it becomes $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}$ &c. the ſum of which is $\frac{\pi^2}{12}$, as we have ſeen; and thus one caſe is obtained in which the ſum of that ſeries is expreſſed by the ſquare of an arc of a circle. But, if I miſtake not, the ingenuity of mathematicians has hitherto diſcovered no other caſe, that of $x = 1$ excepted, in which the ſum of the ſeries $x - \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} +$ &c. can be exhibited by means of arcs of a circle alone, without logarithms.

rithms. Mr. Landen, indeed, has ſucceeded in determining the ſum of the ſeries $x - \frac{x^2}{2^2} + \frac{x^3}{3^2} -$ &c. and the ſums of the kindred ſerieſes $x + \frac{x^2}{2^2} + \frac{x^3}{3^2} +$ &c. and $x + \frac{x^3}{3^2} + \frac{x^5}{5^2} +$ &c. in ſome other particular inſtances by means of circular arcs and logarithms; and theſe caſes I ſhall now proceed to notice, although they ſeem hardly to fall within the ſcope of the queſtion.

For the ſake of abridging language, I ſhall uſe the following notations: *viz.*

the ſymbol $F(x)$ for the ſeries $x + \frac{x^2}{2^2} + \frac{x^3}{3^2} +$ &c.

the ſymbol $f(x)$ for the ſeries $x - \frac{x^2}{2^2} + \frac{x^3}{3^2} -$ &c.

and the ſymbol $\varphi(x)$ for the ſeries $x + \frac{x^3}{3^2} + \frac{x^5}{5^2} +$ &c.

Since $\int \frac{dx}{1 \mp x} = x \pm \frac{x^2}{2} + \frac{x^3}{3} \pm \frac{x^4}{4} +$ &c. therefore $\int \frac{dx}{x} \int \frac{dx}{1 \mp x} = x \pm \frac{x^2}{2^2} + \frac{x^3}{3^2} \pm \frac{x^4}{4} +$ &c: theſe formulas, which are alluded to in the queſtion, belong to De Lorgna: by putting log. $\frac{1}{1-x}$ for $\int \frac{dx}{1-x}$ and log. $(1+x)$ for $\int \frac{dx}{1+x}$, and attending to the meaning of the ſymbols juſt explained, we get

$$\int \frac{dx}{x} \log. \frac{1}{1-x} = F(x)$$

$$\int \frac{dx}{x} \log. (1+x) = f(x)$$

and, adding theſe expreſſions,

$$\int \frac{dx}{x} \log. \frac{1+x}{1-x} = 2\varphi(x).$$

Now, let $x + y = 1$; then $y = 1 - x$; and $-\int \frac{dx}{x} \log. y = -\int \frac{dx}{x} \log. (1-x) = F(x)$: in like manner, $-\int \frac{dy}{y} \log. x = F(y)$; therefore, $F(x) + F(y) = -\int \left\{ \frac{dx}{x} \log. y + \frac{dy}{y} \log. x \right\} = C - \log. x \times \log. y$: to determine C, let $x = 0$, then

$F(x) = 0$, $\log. x \times \log. y = 0$, and $F(y) = F(1) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \&c. = \frac{\pi^2}{6}$: therefore,

$$F(x) + F(1 - x) = \frac{\pi^2}{6} - \log. x \times \log. (1 - x) \ldots\ldots.(1).$$

Let $x = 1 - x$, or $x = (\frac{1}{2})$, then $F(\frac{1}{2}) = \frac{1}{2} + \frac{1}{2^2}(\frac{1}{2})^2 + \frac{1}{3^2}(\frac{1}{2})^3 + \frac{1}{4^2}(\frac{1}{2})^4 + \&c. = \frac{\pi^2}{12} - \frac{1}{2}\log.^2 2.$

Again, let $y = \frac{x}{1 + x}$: then $\frac{dy}{y} = \frac{dx}{x} - \frac{dx}{1 + x}$, and $1 - y = \frac{1}{1 + x}$; therefore $\frac{dy}{y} \log. \frac{1}{1 - y} = \frac{dx}{x} \log. (1 + x) - \frac{dx}{1 + x} \log. (1 + x)$; and, by integrating $F(y) = f(x) - \frac{1}{2}\log.^2 (1 + x)$, that is,

$$f(x) - F\left(\frac{x}{1 + x}\right) = \frac{1}{2}\log.^2 (1 + x) \ldots\ldots\ldots.(2).$$

By the formula (1), we get $F\left(\frac{1}{1 + x}\right) + F\left(\frac{x}{1 + x}\right) = \frac{\pi^2}{6} - \log. \frac{1}{1 + x} \times \log. \frac{x}{1 + x}$: and by combining this equation with the formula (2), we get

$$f(x) + F\left(\frac{1}{1 + x}\right) = \frac{\pi^2}{6} - \frac{1}{2}\log.^2 (1 + x) + \log. (1 + x) \times \log. x \ldots(3).$$

Let $a = \frac{1}{1 + a}$; then $a^2 + a = 1$, and $a = \frac{\sqrt{5} - 1}{2}$: put a for x in the formula (3), then

$$f(a) + F(a) = \frac{\pi^2}{6} - \frac{3}{2}\log.^2 a.$$

Put a for x in the formula (2), and becauſe $a^2 = \frac{a}{1 + a}$, therefore

$$f(a) - F(a^2) = \frac{1}{2}\log.^2 a.$$

But, from the nature of the functions $f(a)$ and $F(a)$, it is obvious that we have $F(a) - \frac{1}{2}F(a^2) = f(a)$; and thus three equations are obtained that are ſufficient to determine the quantities $F(a)$, $f(a)$, and $F(a^2)$; put $b = a^2 = \frac{3 - \sqrt{5}}{2}$; and we thus get

F (a)

$$F(a) = a + \frac{a^2}{2^2} + \frac{a^3}{3^2} + \&c. = \frac{\pi^2}{10} - \log.^2 a$$

$$f(a) = a - \frac{a^2}{2^2} + \frac{a^3}{3^2} - \&c. = \frac{\pi^2}{15} - \tfrac{1}{2}\log.^2 a$$

$$F(a^2), \text{ or } F(b) = b + \frac{b^2}{2^2} + \frac{b^3}{3^2} + \&c. = \frac{\pi^2}{15} - \tfrac{1}{4}\log.^2 b$$

where $a = \frac{\sqrt{5} - 1}{2}$ and $b = \frac{3 - \sqrt{5}}{2}$.

Suppoſe $y = \frac{1 - x}{1 + x}$: then $-\int\frac{dx}{x}\log.\, y = \int\frac{dx}{x}\log.\frac{1+x}{1-x} = 2\varphi(x)$: and becauſe $x = \frac{1-y}{1+y}$, therefore, in like manner, $-\int\frac{dy}{y}\log.\, x = 2\varphi(y)$: therefore,

$$\varphi(x) + \varphi(y) = -\tfrac{1}{2}\int\left\{\frac{dx}{x}\log.\, y + \frac{dy}{y}\log.\, x\right\} = c - \tfrac{1}{2}\log.\, x \times \log.\, y:$$

but when $x = 0$, then $\varphi(x) = 0$, $\log.\, x \times \log.\, y = 0$, and $\varphi(y) = \varphi(1) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \&c. = \frac{\pi^2}{8} = c$: therefore,

$$\varphi(x) + \varphi\left(\frac{1-x}{1+x}\right) = \frac{\pi^2}{8} - \tfrac{1}{2}\log.\, x \times \log.\frac{1-x}{1+x} \;\ldots\ldots(4)$$

let $e = \frac{1-e}{1+e}$; then $e^2 - 2e = 1$, and $e = \sqrt{2} - 1$: and by putting $x = e$, the formula (4) gives

$$\varphi(e) = e + \frac{e^3}{3^2} + \frac{e^5}{5^2} + \&c. = \frac{\pi^2}{16} - \tfrac{1}{4}\log.^2 e.$$

Again, a being $= \frac{\sqrt{5} - 1}{2}$, we have obtained above, $f(a) + F(a) = \frac{\pi^2}{6} - \tfrac{3}{2}\log.^2 a$: but $f(a) + F(a) = 2\varphi(a)$: therefore,

$$\varphi(a) = a + \frac{a^3}{3^2} + \frac{a^5}{5^2} + \&c. = \frac{\pi^2}{12} - \tfrac{3}{4}\log.^2 a.$$

Laſtly, let $d = \frac{1-a}{1+a} = \sqrt{5} - 2$: then, by ſubſtituting the value of $\varphi(a)$ juſt obtained, in the formula (4), we obtain

$$\varphi(d) = d + \frac{d^3}{3^2} + \frac{d^5}{5^2} + \&c. = \frac{\pi^2}{24} + \tfrac{3}{4}\log.^2 a - \tfrac{1}{2}\log.\, a \times \log.\, d.$$

The formulas (2) and (3) deſerve notice, becauſe the computation of F (x), when x is leſs than 1, is, by them, reduced to the computation of $f(x)$, when x is leſs than 1. And, becauſe $x = \frac{1+x}{1-\frac{1}{x}}$, therefore log. $x =$ log. $(1+x) -$ log. $\left(1+\frac{1}{x}\right)$: therefore $\frac{dx}{x}$ log. $(1-x) - \frac{dx}{x}$ log. $\left(1+\frac{1}{x}\right) = \frac{dx}{x}$ log. x: therefore $f(x) + f\left(\frac{1}{x}\right) = C + \frac{1}{2}\log.^2 x$: to determine C put $x = 1$, then $C = 2f(1) = \frac{\pi^2}{6}$; therefore,

$$f(x) + f\left(\frac{1}{x}\right) = \frac{\pi^2}{6} + \tfrac{1}{2}\log.^2 x \ldots\ldots\ldots\ldots (5).$$

By this formula, the computation of $f(x)$ when x is greater than 1, is reduced to the computation of $f(x)$ when x is leſs than 1. Thus the computation of F (x) when x is leſs than 1, and of $f(x)$ in all caſes whatſoever, is reduced to the computation of $f(x)$ when x is leſs than 1. Now this may be accompliſhed by a converging ſeries as follows:

Let $z =$ log. $\sqrt{(1+x)}$ and let c be the number whoſe hyperbolic logarithm is 1: then $c^{2z} = 1 + x$: therefore $dx = 2dz \,.\, c^{2z}$, $x = c^{2z} - 1$, and $\frac{dx}{x}$ log. $(1+x) = \frac{4zc^{2z} \,.\, dz}{c^{2z}-1} = \frac{4zc^{z} \,.\, dz}{cz - c^{-z}} = 2zdz + 2dz \times \frac{z(c^z + c^{-z})}{c^z - c^{-z}}$. Let $\frac{z(c^z + c^{-z})}{c^z - c^{-z}} = \dfrac{1 + \frac{z^2}{1.2} + \frac{z^4}{1.2.3.4} + \frac{z^6}{1.2.3.4.5.6} \text{ \&c.}}{1 + \frac{z^2}{1.2.3} + \frac{z^4}{1.2.3.4.5} + \frac{z^6}{1.2.3.4.5\,6.7} \text{ \&c.}}$ be developed into a ſeries $1 + A^{(2)}z^2 - A^{(4)}z^4 + A^{(6)}z^6 -$ &c. then by ſubſtituting, and taking the fluents, we get

$$f(x) = z^2 + 2 \times \left\{ z + A^{(2)} . \frac{z^3}{3} - A^{(4)} . \frac{z^5}{5} + A^{(6} . \frac{z^7}{7} - \&c. \right\}.$$

XXVII. QUESTION 177, *by* MARLOVIENSIS.

It is required to ſhew a ſimple practical method (capable of demonſtration), of putting a globe or ſphere into perſpective; having given the magnitude and ſituation of the ſphere, the height of the eye above the ground plane, and its diſtance from the plane of projection, which is, as uſual, ſuppoſed to be perpendicular to the horizon, alſo to ſhew what ſituation the globe muſt have, that the contour of its perſpective may be any propoſed conic ſection ?

SOLUTION, *by Mr.* LOWRY.

The rays proceeding from that part of the globe which is viſible to the eye, will evidently form a right cone whoſe vertex is at the eye, and its baſe a circle made by the contact of the ſphere wtth the viſual rays that form the conic ſurface; and the contour of the perſpective repreſentation of the globe, on the picture, will be the figure made by the interſection of the plane of the picture with the conic ſurface; and will be one or other of the conic ſections according to the relative poſitions of the cone and plane. Conceive a plane to paſs through the eye in a direction parallel to the plane of projection, or plane of the picture : this plane is generally called the directing plane: then, when the globe is placed either on the contrary ſide of the picture to the eye, or between the picture and the directing plane, but ſo as not to touch that plane, the cone of rays will be interſected on all ſides by the plane of the picture, and the figure will be an ellipſe, except in that particular caſe when the baſe of the cone is parallel to the plane of the picture, and then the figure will be a circle. Again, when the globe touches the directing plane on the ſide next to the picture, a ſide of the cone will be in the directing plane, and conſequently will be parallel to the plane of the picture, and the ſection will be a parabola: but when the globe is placed ſo as to be cut by the directing plane, it is evident that the part of the globe on the ſide next to the picture will only be viſible to the eye, and the figure will be a hyperbola. Hence, we ſee, that the contour of the perſpective of the globe, will be an ellipſe, when the globe is placed without the directing plane; a parabola when it touches the directing plane, on the ſide next to the picture; and a hyperbola when the globe is cut by the directing plane.

Let E (fig. 93, 94, pl. 4.) be the poſition of the eye, AB*ab* the plane of projection, or, as it is uſually called, the plane of the picture, and H the centre of the ſphere. Alſo, let EC be drawn perpendicular to the plane AB*ab*, and conceive a plane to paſs through the eye, the centre of the picture, and the centre of the ſphere; or through the points E, C, H, meeting the plane of the picture in the line CI, and the horizontal plane paſſing through H in the line HI. Then, ſince the relative poſitions of the globe, the eye, and the plane of the picture are given; the diſtances IH, IC, and EC are given. Let the rays EG, EF touch the circle FGW, which is the ſection of the ſphere by the plane ECH, in G and F, and meet CI in *g* and *f*, and join GF; then GF is evidently the diameter of the baſe of the viſual cone, and gf its perſpective repreſentation on the plane of the picture. And ſince gf paſſes through the axis of the cone, and alſo through the point C, where a perpendicular from the vertex of the cone meets the interſecting plane, it is eaſy to prove that when the cone is right, gf is one of the axes of the ſection, namely, the greater axis when the figure is an ellipſe, and the greater, or leſs axis, according as the cone is acute or obtuſe-angled, when the figure of the ſection is a hyperbola. It is alſo evident that the other axis of the ſection will be parallel to the baſe of the cone.

Biſect gf in S the centre of the ellipſe or hyperbola, and draw MSN parallel to GF, meeting EG in N and EF in M; and let PSQ, drawn perpendicular to gf, be the other axis; then the points P, Q, N, M being in the conical ſurface, and in a plane parallel to the baſe, they are in the circumference of a circle, and $MS\cdot SN = PS\cdot SQ =$ to the ſquare of the ſemi-axis SP or SQ. From hence we derive the following practical conſtruction: take a line $ki =$ to the diſtance HI, and with a diſtance equal to the radius of the ſphere, and centre k, deſcribe a circle. Make ic perpendicular to ik and equal to the diſtance IC: and make ce perpendicular to ic and $=$ to CE, the diſtance of the eye from the plane of the picture: draw the tangents eG', eF' meeting ic in g' and f'; then it is evident that the diſtances if', cg' are reſpectively equal to the diſtances If, Cg, and therefore the axis fg may be determined by making $If = if'$, and $Cg =$ to cg'. Let $f'g'$ be biſected in s, and through s draw mn parallel to $F'G'$; then ms, sn are reſpectively equal to MS, SN, and therefore the rectangle $ms.sn$ is $=$ to the ſquare of the ſemi-axis PS or SQ, hence if PS or SQ be taken each equal to a mean proportional between ms and sn, we have the two axes of the ſection and the perſpective curve may be deſcribed by known methods.

The figure is drawn only for the caſe when the contour of the perſpective of the globe is an ellipſe, but the above conſtruction is equally applicable when the contour is an hyperbola, as will be very evident to any one who will take the trouble to draw the figure.

Inſtead of finding the axis PQ, we might have found an ordinate OL, paſſing through the point O where the axis fg meets EH the axis of the cone. For if GF meet EH in K, it is evident (ſince OL is parallel to the baſe of the cone) that $EO : LO :: EK$: the radius of the baſe of the cone; and ſince EO, EK (or eo, eK') and the radius KF are given; it is evident that LO is given, and it may be found by drawing ol parallel to $G'F'$; for then OL is $= ol$. This method of conſtruction is immediately applicable when the contour of the perſpective of the globe is a parabola.

XXVIII. QUESTION 178, *by* MECHANICUS.

If two given bodies, A and B, connected by a ſtring, be ſupported on an inclined plane, by means of the ſtring paſſing through a ring at A, and any given force be impreſſed on the body A, in a horizontal direction; it is required to determine the nature of the curve deſcribed by the body, and the circumſtances of the motion?

SOLUTION, *by* MECHANICUS, *the Propoſer.*

Let CDEF (fig. 186, pl. 4.) be the inclined plane, n the ſine of its inclination to the horizon, R the poſition of the ring, or the point where the body A is placed at the commencement of the motion; A and B the poſition of the bodies at the end of any indefinite time t, and RAa a portion of the curve deſcribed by the body A. Alſo let QR, parallel to the horizon, be the direction of the impulſe which communicates motion to the bodies, and draw AQ perpendicular to RQ, or parallel to the ſtring RB.

Let Aa be an indefinitely ſmall portion of the curve, draw am perpendicular to QAS, and join AR, aR; with the diſtance AR and centre R deſcribe the ſmall arc An and take Bb = an.

Put l = the length of the ſtring,
g = $32\frac{1}{6}$ feet, the meaſure of the accelerative force of gravity,
x = AR, and then BR is = $l - x$,
v = the velocity in the curve at A,
u = the velocity in the direction AR, or the velocity with which the bodies approach to or recede from the ring,
y = AQ, z = RQ, $\dot{s}$ = Aa, $\dot{x}$ = an = Bb, $\dot{y}$ = am.

The impulſe given to the body A, when at R, cauſes it to recede from the ring and draw up the body B on the inclined plane in the direction BR: the motive force of A in the direction AS is $= ng \times \text{A}$, which being reſolved into the direction RA is $= ng \times \text{A}\,\frac{\dot{y}}{\dot{x}}$; and ſince B acts in oppoſition to this force, its effect in the direction of the ſtring muſt be ſubtracted from $ng \times \text{A}\,\frac{\dot{y}}{\dot{x}}$ to obtain the motive force, which, upon the whole, urges A or B towards the ring; but the motive force of B, or the force that urges it to deſcend in the direction BR is $= ng \times \text{B}$, therefore $ng \times (\text{A}\,\frac{\dot{y}}{\dot{x}} - \text{B}) = ng \times (\frac{\text{A}\dot{y} - \text{B}\dot{x}}{\dot{x}})$ is the motive force in the direction of the ſtring. And this force reſolved into the direction Aa is $= ng \times (\frac{\text{A}\dot{y}}{\dot{x}} - \text{B}) \times \frac{\dot{x}}{\dot{s}} = ng \times (\frac{\text{A}\dot{y} - \text{B}\dot{x}}{\dot{s}})$ the motive force in the direction Aa of the curve,

curve. Theſe forces divided by the inertia of the maſs moved, are equal to the reſpective accelerative forces in the directions AR, A*a*.

The inertia of B is evidently equal to its own maſs B; but ſince A and B move with different velocities, or are differently accelerated, the inertia of A cannot be equal to its own maſs but to ſome other quantity. Let this quantity be M, that is, let M be a maſs ſuch, that when placed at B, and the maſs A removed, it may reſiſt the communication of motion in the ſame manner as the maſs A when it moves in its proper path.

Now the motive force of A, in the direction A*a*, being $ng \times \left(\frac{A\dot{y} - B\dot{x}}{\dot{s}}\right)$, the accelerative force in that direction is $ng \times \left(\frac{A\dot{y} - B\dot{x}}{\dot{s}A}\right)$ if A only be conſidered as the maſs moved; and the motive force of M (or B), in the direction BR, being $ng \times \left(\frac{A\dot{y} - B\dot{x}}{\dot{x}}\right)$, the accelerative force in that direction is $ng \times \left(\frac{A\dot{y} - B\dot{x}}{M\dot{x}}\right)$ if M only be conſidered as the maſs moved: But A deſcribes the ſpace $\dot{s}$ with the velocity v, in the ſame time that M deſcribes the ſpace $\dot{x}$ with the velocity u, and the accelerative force being equal to the fluxion of the velocity divided by the fluxion of the time, the accelerative forces of A and M will be as the fluxions of the velocities, that is, $ng \times \left(\frac{A\dot{y} - B\dot{x}}{A\dot{s}}\right) : ng \times \left(\frac{A\dot{y} - B\dot{x}}{M\dot{x}}\right) :: \dot{v} : \dot{u}$; therefore $M = \frac{A\dot{s}\dot{v}}{\dot{x}\dot{u}}$; and $B + \frac{A\dot{s}\dot{v}}{\dot{x}\dot{u}}$ = the whole inertia of the maſs.

Conſequently, $ng \times \left(\frac{A\dot{y} - B\dot{x}}{\dot{x}}\right) \div \left(B + \frac{A\dot{s}\dot{v}}{\dot{x}\dot{u}}\right) = ng \times \frac{(A\dot{y} - B\dot{x})\,\dot{u}}{B\dot{u}\dot{x} + A\dot{v}\dot{s}}$ is the accelerative force in the direction AR or B*b*, which, by dynamics, is = to $\frac{u\dot{u}}{\dot{x}}$; therefore $ng\,(A\dot{y} - B\dot{x}) = Bu\dot{u} + \frac{A\dot{s}u\dot{v}}{\dot{x}}$, or ſince $\frac{\dot{s}}{\dot{x}} = \frac{v}{u}$

$$ng\,(A\dot{y} - B\dot{x}) = Bu\dot{u} + Av\dot{v} \ldots\ldots\ldots\ldots (1)$$

and taking the fluents

$$2ng\,(Ay - Bx) = Bu^2 + Av^2 \ldots\ldots\ldots\ldots (2),$$

Now, beſides the force which acts in the direction of the ſtring, there is another force which diſturbs the motion of the bodies by acting in a direction perpendicular to the radius vector RA, for the motive force of A in the direction QSA is $ng \times A$, which being reſolved into a force perpendicular to RA is $ng \times A \times$ ſin. $QAR = ng \times A \times \frac{z}{x} = A \times f$ (ſuppoſing $ng \times \frac{z}{x} = f$).

To compute the effect of this force on the motion of the bodies; let w be the velocity of A, in a direction perpendicular to the radius vector AR, and let p be the value of the perpendicular drawn from R to the tangent to the curve at A: then the velocity in the direction of the tangent, or element Aa of the curve, will be $\frac{w}{\text{ſin. PAR}} = \frac{wx}{p}$. This velocity would continue invariable, if the forces were to ceaſe to act on the body A, and its fluxion $\frac{w\dot{x} + \dot{w}x}{p}$ would then be $= 0$; or $\dot{w}$ would be $= - \frac{w\dot{x}}{x}$; but when the forces continue to act, the fluxion of the velocity is $\dot{w}$, therefore $\dot{w} + \frac{w\dot{x}}{x}$ (or the exceſs of the fluxion of the velocity when the forces act, above the fluxion of the velocity when the forces ceaſe to act,) will be equal to the fluxion of the velocity generated or deſtroyed by the action of the motive force $A \times f$ only. But the fluxion of the velocity is $=$ to the element of the time $\dot{t}$ $\left(\text{or } \frac{\dot{x}}{u}\right)$ multiplied by the accelerative force, that is,

$$\dot{w} + \frac{w\dot{x}}{x} = \frac{\dot{x}}{u} \times f, \text{ or } f = \left(\dot{w} + \frac{w\dot{x}}{x}\right) \times \frac{u}{\dot{x}},$$

But it is evident that w is $= \sqrt{(v^2 - u^2)}$, therefore $\dot{w} = \frac{v\dot{v} - u\dot{u}}{\sqrt{(v^2 - u^2)}}$; and putting theſe values in the expreſſion for f, we have

$$f = \left(\frac{v\dot{v} - u\dot{u}}{\sqrt{(v^2 - u^2)}} + \sqrt{(v^2 - u^2)} \times \frac{\dot{x}}{x}\right) \times \frac{u}{\dot{x}},$$

$$\text{or, } ngz\ (= fx) = \frac{(v\dot{v} - u\dot{u})\ xu}{\dot{x}\sqrt{(v^2 - v^2)}} + u\sqrt{(v^2 - u^2)} \ldots\ldots (3).$$

From the equations (1) (2) (3), and recollecting that $\dot{s} = \sqrt{(\dot{y}^2 + \dot{z}^2)}$, $\frac{\dot{s}}{\dot{x}} = \frac{v}{u}$, and $z^2 = x^2 - y^2$, we may determine the circumſtances of the motion, and alſo the relation between y and z, the rectangular co-ordinates of the curve; but the expreſſions are too complicated to admit of any elegant concluſion.

XXIX.

XXIX. QUESTION 179, *by Mr.* WALLACE, *R. M. College.*

Suppoſe two particles of matter p and q, to be connected by a ſtring pAq, and to reſt in equilibrio upon two inclined planes AB, AC, the ſtring being ſuppoſed to paſs over the common ſection of the planes at the point A; then, if a circle be deſcribed through the points p, A, q, and the particles without changing their poſition are diſengaged from the ſtring pAq, and connected by another paq, which paſſes over a peg faſtened at any point a in the circumference of the circle, the particles p and q will ſtill reſt in equilibrio upon the planes. Required the demonſtration?

SOLUTION, *by Mr.* THOMAS BAZLEY, *Bolton.*

Fig. 87, pl. 4. Draw the horizontal line pQ cutting AC in Q; then, by the property of the inclined plane (when the ſtring paſſes over the common ſection of the planes) rad. 1 : ſin. ApQ :: p : p × ſin. ApQ = the force ſuſtaining p, and in like manner q × ſin. AQp = the force ſuſtaining q; whence by the queſtion ſin. $ApQ \times p$ = ſin. $AQp \times q$.

Now when the ſtring paſſes over a, we ſhall have, (by cor. 1. prop. 15, Emerſon's Mechanics, 8vo.), coſ. apA : ſin. ApQ :: p : (ſin. ApQ ÷ coſ. apA) × p = the force of p in the direction ap; and for the ſame reaſon, (ſin. AQp ÷ coſ. Aqa) × q = the force of q in the direction aq. But it was ſhewn that ſin. $ApQ \times p$ = ſin. $AQp \times q$, and, by the property of the circle, the angle Apa is = to the angle Aqa; therefore $\frac{\text{ſin. } ApQ}{\text{coſ. } Apa} \times p = \frac{\text{ſin. } AQp}{\text{coſ. } Aqa} \times q$; conſequently p and q ſuſtain each other at a.

And thus the queſtion was anſwered by Meſſrs. Cunliffe, Nicholſon, *and* Simpſon.

XXX. PRIZE QUESTION 180, *by* GEOMETRICUS.

If the oppoſite ſides of a hexagon, inſcribed in a circle, be produced till they meet, the three points of interſection will be in the ſame ſtraight line. Required the demonſtration?

FIRST SOLUTION, *by the late Earl* STANHOPE.

(From the Appendix to Schawb's edition of Euclid's Data.)

Fig. 88, pl. 4. Let the oppoſite ſides of the hexagon AFEDCB, when produced, meet in the points G, H, I.

Draw BL parallel to EI, and FM parallel to CG; and through F, B, draw LP, MN; join PH, NH, and draw FC and BE.

Becauſe, by conſtruction, BL is parallel to ED, the arc BCD = LFE, and the angle BCD = LFE, and their ſupplements or the angles PCH, PFH are equal; and ſince the angle FQP = CQH, the remaining angles of the triangles FQP, CQH are equal, or the angle CHQ = FPQ. Therefore the quadrilateral FCHP may be circumſcribed by a circle. And in the ſame manner it is proved that a circle will circumſcribe the quadrilateral BEHN.

Hence the angle CFH = CPH : but the angles CFH, CDN are equal; therefore the angle CPH = CDN. In like manner it is proved that the angle EDP = ENH, therefore EI is parallel to PH, and DG to HN, and conſequently DNHP is a parallelogram: therefore the angle QPH = RNH.

Again, the angle AFM = AGC, and AFM = ABM, therefore ABM = AGC. But becauſe the angle ABM = IBN, the angle IBN = AGC. And in the ſame manner it is proved that the angle GFP = BIN. Therefore the triangles FGP, BIN are equiangular; conſequently the ſupplement of the angle BNI or the angle BNR is equal to FPQ the ſupplement of FPG. And ſince the angle QPH = RNH, therefore FPQ + QPH or FPH = BNR + RNH. But HBN = HEN, and HEN = FHP, therefore the triangles BNH, FPH are ſimilar. And becauſe the triangles FGP, BIN are alſo ſimilar, we have

GP : PF :: BN : NI; and PH : PF :: BN : HN;
therefore (*ex æq. perturb.*) GP : PH :: HN : NI:
whence (by comp.) GP : PH :: GP + HN (or GD) : PH + NI or DI:
therefore GHI is a ſtraight line.

Second Solution, *by Mr.* Ivory.

Fig. 89, pl. 4. Let the oppoſite ſides of the hexagon meet in the points G, H, I.

Join two of the oppoſite angles of the hexagon, as A and D; about the triangle ADI deſcribe a circle, and let that circle meet GA and GD (or theſe lines produced) in K and L; and join FC, KI, LI, KL. Becauſe the quadrilateral ADIK is in a circle, therefore the angle AKI = ADE: and becauſe the quadrilateral FADE is in a circle, therefore the angle ADE = GFE: therefore the angle AKI = GFE, and KI is parallel to FH. In like manner, on account of the circles, the angle AIL = ADL = CBI; and LI is parallel to HB. And, again, the angle KLD = DAF = FCD; and KL is parallel to FC. Becauſe FC is parallel to KL, therefore

GF : FC :: GK : KL.

And, becauſe the ſides of the triangle HFC have been ſhewn to be parallel to the ſides of the triangle IKL, each to each, therefore

FC : FH :: KL : KI.

Therefore, *ex æquo*, GF : FH :: GK : KI.

Therefore G, H, I are in one ſtraight line.

Third Solution, *by Mr.* Lowry.

Let ABCDEFA (fig. 90, pl. 4.) be the inſcribed hexagon; G the interſection of the oppoſite ſides BC, EF; H the interſection of the oppoſite ſides AB, DE; and I the interſection of the oppoſite ſides CD, AF; then the points G, H, I are in a ſtraight line.

Draw HK parallel to BC meeting CD produced in K, and GL parallel to AB meeting IF in L. Join AK, LC, AD and CF. Then, becauſe HK is parallel to BC, the angle BCI (or

BAD, Euc. iii. 22.) is = to the angle HKD; therefore the points A, D, K, H are in the circumference of the ſame circle, and the angle ADH = to the angle AKH.

Again, becauſe GL is parallel to BA, the angle GLF is = to the angle HAF or *(ibid.)* to the angle BCF; therefore the points G, L, C, F are in the ſame circle, and the angle GCL = the angle GFL = the angle ADH or AKH; therefore LC is parallel to AK, and conſequently the triangle ILC is equiangular to the triangle IAK, and the triangle GLC equiangular to the triangle HAK; therefore LG : AH :: LC : AK :: IL : IA; and ſince the points I, L, A are in a ſtraight line, it is evident that the points I, G, H are alſo in a ſtraight line.

The propoſition may be enunciated more generally as follows:

Let two chords AB, DE (fig. 91, pl. 4.) be inſcribed in a circle, interſecting at H (either within or without the circle), and from two of the terminations B, D (or B, E) let two ſtraight lines be inflected to any point C in the circle, and from the other terminations A, E (or A, D) let two other ſtraight lines be inflected to any point F in the circle, meeting the former inflected lines in G and I; then the points G, H, I will be in a ſtraight line.

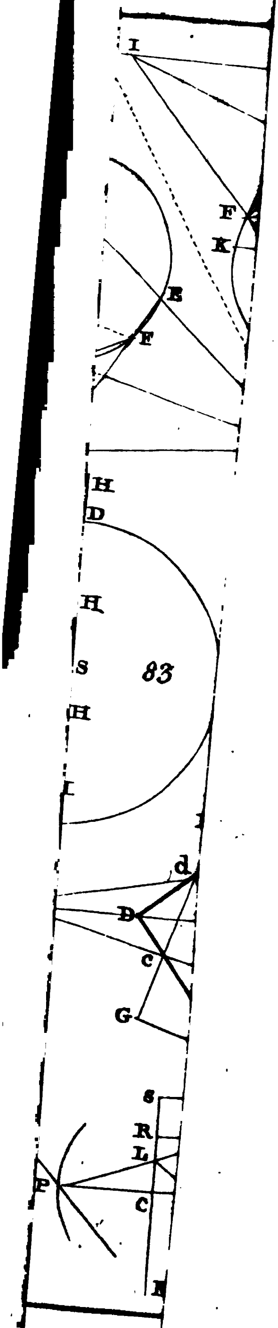
If any two of the points in the circle coincide, a tangent may be ſubſtituted for the inflected line, and the propoſition will ſtill be true, and the preceding demonſtration will hold good.

The propoſition thus generalized, is alſo true in any of the conic ſections, as was, I believe, firſt diſcovered by Maclaurin. It is alſo true when the figure is inſcribed in a ſmall circle on the ſurface of a ſphere; arches of great circles being ſubſtituted in place of ſtraight lines.

FOURTH SOLUTION, *by Mr.* RICH. NICHOLSON, *Liverpool.*

Let ABCDEFA (fig. 92, pl. 4.) be any irregular hexagon inſcribed in a circle; produce the ſides AB, ED and BC, EF to interſect in G and H; draw GH, which produce to meet CD in I, and join IF, FA: then, if the propoſition be true, FA is a continuation of IF. To demonſtrate this, draw BK parallel to GI, and KL through F to meet GI in L; join F, C and L, C: then, the angles GHB, KBH and LFC being equal, the points F, C, H, L are in a circle; therefore the angles HLC, HFC and GDC are equal and the points G, C, D, L are in a circle; wherefore the angles KFA, KBA and LGB are equal, and the points A, G, L, F are alſo in a circle. Now, if FA be not a continuation of IF let FM be a continuation of it meeting the circle in M; then IG × IL = IC × ID = IM × IF, therefore the points G, F, L, M are in the ſame circle as G, L, F, A; therefore, joining M, B; M, G, the angle, MGA, is equal to the angle MFA, that is, to the angle MBA, which is abſurd. Wherefore FA is a continuation of IF, and the propoſition of Geometricus true.

Mr. IVORY *is requeſted to ſend to Mr.* GLENDINNING'S *for the Medal for ſolving the Prize Queſtion.*



UNIV. OF MICH.

ARTICLE III.

Solutions to Questions propoſed in Number VI.

I. QUESTION 181, *by* YANTO.

Bacchus caught Silenus aſleep by the ſide of a full caſk, he ſeized the opportunity of drinking, which he continued for two-thirds of the time that Silenus would have taken to empty the whole caſk. After that, Silenus awakes, and drinks what Bacchus had left. Had they drank both together, it would have been emptied two hours ſooner, and Bacchus would have drank only half of what he left for Silenus. Required the time in which they would empty the caſk ſeparately?

SOLUTION, *by* YANTO, *the Propoſer.*

Let x denote the number of hours in which Bacchus alone would empty the cask, and y the number of hours in which Silenus would do the ſame. Then if the content of the cask be denoted by unity, $\frac{1}{x}$ and $\frac{1}{y}$ will expreſs the reſpective quantities drank by Bacchus and Silenus in an hour; and by the queſtion, $\frac{2}{3}y$ is the time Bacchus continued drinking while Silenus was aſleep; therefore, by the queſtion, we ſhall alſo have $x : \frac{2y}{3} :: 1 : \frac{2y}{3x} =$ the quantity drank by Bacchus whilſt Silenus ſlept; and $1 - \frac{2y}{3x} = \frac{3x - 2y}{3x} =$ the quantity which Silenus drank: and therefore, $\frac{1}{y} : 1 :: \frac{3x - 2y}{3x} : \frac{y(3x - 2y)}{3x}$ $=$ the time Silenus was drinking: conſequently, $\frac{2}{3}y + \frac{y(3x - 2y)}{3x} = \frac{5xy - 2y^2}{3x}$ will expreſs the whole time of drinking the cask. Again, $\frac{1}{x} + \frac{1}{y} : 1 :: 1 : \frac{xy}{x + y} =$ the time in which both together might empty the caſk; and therefore by the queſtion,

$$\frac{5xy - 2y^2}{3x} = \frac{xy}{x + y} + 2.$$

Alſo, $\frac{1}{x} + \frac{1}{y} : 1 :: \frac{1}{x} : \frac{y}{x + y}$ = Bacchus's ſhare, had they drank both together; which, by the queſtion, muſt be equal to half of what was left for Silenus; that is,

$$\frac{3x - 2y}{3x} = \frac{2y}{x + y}, \text{ or } 3x^2 - 5xy = 2y^2:$$

Hence, $x = 2y$, or $-\frac{1}{3}y$: but as the queſtion does not admit of x or y being negative, the only value of any use, is $x = 2y$, by means whereof, exterminate x out of the equation $\frac{5xy - 2y^2}{3x} = \frac{xy}{x + y} + 2$, and it becomes $\frac{4y}{3} = \frac{2y}{3} + 2$: whence $y = 3$ and $x = 2y = 6$.

And thus the Queſtion was anſwered by Meſſrs. Amicus, Cunliffe, O'Riordan, Toplis, *and* Vinoſus.

II. QUESTION 182, *by* AMICUS.

Two given beams, or rods, loaded with weights, being placed on a horizontal plane, ſo that their lower extremities may be oppoſed to each other, and the other extremities be ſupported by two parallel vertical planes; it is required to find their poſition when they are in equilibrio, and the preſſure exerted againſt each plane?

FIRST SOLUTION, *by Mr.* CUNLIFFE, *R. M. College.*

Let AB, (fig. 96, pl. 5.) repreſent the horizontal plane upon which the two beams or rods are placed; AM and BP the two vertical planes; NM and NP the two rods or beams oppoſed to each other at their lower ends N, and the other ends leaning againſt the vertical planes AM and BP.

Let G, g repreſent the reſpective centres of gravity of the beams NM and NP: draw MR, PS parallel to AB and RGH, SgL perpendicular thereto, and join RN, SN. Let w denote the weight of the beam NM, and w' the weight of the beam NP.

Then by the corollary to Prop. 63, Emerſon's Mechanics, RH : HN :: w : $(\text{HN} \div \text{RH}) \times w$ = the force exerted by the beam NM in the direction AN: and for the ſame reaſon, $(\text{LN} \div \text{SL}) \times w'$ = the force exerted by the beam NP in the direction BN: and theſe oppoſite forces when the beams are in equilibrio, muſt counteract or be equal to each other; that is

$$(\text{HN} \div \text{RH}) \times w = (\text{LN} \div \text{SL}) \times w',$$

or $$\text{HN} \times \text{SL} \times w = \text{LN} \times \text{RH} \times w'.$$

And by reaſon of the parallels HG, AM; Lg, BP,

$$\text{NG} : \text{NM} :: \text{NH} : \text{NA} = \text{NM} \times \text{NH} \div \text{NG};$$
$$\text{N}g : \text{NP} :: \text{NL} : \text{NB} = \text{NP} \times \text{NL} \div \text{N}g.$$

Put NG $= a$, GM $= b$, Ng $= c$, gP $= d$, NH $= x$, NL $= y$, and AB $= s$. Then GH $= \sqrt{(a^2 - x^2)}$, and gL $= \sqrt{(c^2 - y^2)}$, and from what has been deduced, NA $= (a+b)x \div a$ and NB $= (c+d)y \div c$.

Again, becauſe of the parallels HG, AM; Lg, BP,

NG : NM :: HG : AM $= (a+b)\sqrt{(a^2 - x^2)} \div a =$ RH,

Ng : NP :: Lg : BP $= (c+d)\sqrt{(c^2 - y^2)} \div c =$ SL, by means of which, the equation HN $\times$ SL $\times$ w = LN $\times$ RH $\times$ w' becomes

$$wx(c+d)\sqrt{(c^2 - y^2)} \div c = w'y(a+b)\sqrt{(a^2 - x^2)} \div a,$$

also, NA + NB $= (a+b)x \div a + (c+d)y \div c = s$.

From the two preceding equations, x and y may be found, which will determine the poſition of the beams when in equilibrio. And the preſſure of each beam againſt its reſpective vertical plane, is equal to that exerted by its lower end in an horizontal direction, (cor. prop. 63, Emerſon's Mechanics), and therefore the preſſure againſt each of the vertical planes when the beams are in equilibrio, is expreſſed by $(\text{HN} \div \text{RH}) \times w$. The preſſure againſt the horizontal plane is obviouſly equal to the weight of both beams or $w + w'$.

SECOND SOLUTION, *by the Rev.* I. TOPLIS.

Let AG, MP, (fig. 97, pl. 5,) repreſent the two vertical planes, GM the horizontal one, AH, PH the two beams, D and Q their centres of gravity: let DF and QS, perpendicular to the horizon, repreſent the reſpective weights of the loaded beams. From D draw DE perpendicular, and FE parallel to AH: then DF may be decompoſed into the forces DE, EF. In like manner, let QR perpendicular, and RS parallel to PH, repreſent the force QS, decompoſed into theſe directions. Let AC perpendicular to AG, repreſent the re-action of the plane AG upon the beam AH, which re-action may be decompoſed into two others, AB perpendicular, and BC parallel to AH. In like manner, PN may be decompoſed into PT perpendicular, and TN parallel to PH. In the direction of the beam AH, let

Hc be taken equal to BC + EF, and decompoſed into cd and dH; then Hc repreſents the force which acts upon the beam in the direction AH. Let Ha be the force with which the beam AH preſſes upon the plane GM, in a direction perpendicular to itſelf, which may be decompoſed into the forces Hb parallel, and ba perpendicular to GH, Let He = TN + RS repreſent the force of the beam PH, in the direction PH, which may be decompoſed into Hf parallel, and fe perpendicular to GM; alſo the preſſure Hg, of the end of the beam PH, may be decompoſed into the forces Hh parallel, and hg perpendicular to GM.

If AB = HD × DE ÷ AH, cH = BC + EF, Ha = AD × DE ÷ AH, PT = QR × QH ÷ PH, eH = TN + RS, Hg = QR × QP ÷ PH, and $Hd - Hb = Hf - Hh$, it is evident, from the known principles of Mechanics, that the beams will be in equilibrio.

To calculate this poſition, we may eaſily perceive that the right-angled triangles AGH, ABC, DEF, Hcd and Hab have the angles GAH, BAC, EFD, Hcd and bHa equal, therefore the triangles are ſimilar. Alſo the right-angled triangles PTN, PHM, QRS, Hef and Hgh are ſimilar, for the angles NPT, HPM, QSR, feH, and gHh are evidently equal.

Put $\angle\ GHA = x$, $\angle\ PHM = y$, $DF = a$, the weight of the beam AH; $DA = b$, $DH = c$, $AH = b + c = d$, $QS = m$, the weight of the beam PH; $QH = h$, $QP = k$, $PH = h + k = f$, $GM = g$; then $DE = a \text{ coſ. } x$, and $EF = a \text{ ſin. } x$, $AB = HD \times DE \div AH = \frac{ac}{d} \text{ coſ. } x$, and $BC = \frac{AB}{\text{tan. } x} = \frac{ac}{d} \cdot \frac{\text{coſ. } x}{\text{tan. } x}$; therefore $Hc = BC + EF = \frac{ac}{d} \cdot \frac{\text{coſ. } x}{\text{tan. } x} + a \times \text{ſin. } x$, and $Hd = Hc \cdot \text{coſ. } x = \frac{ac}{d} \cdot \frac{\text{coſ.}^2 x}{\text{tan. } x} + a \text{ ſin. } x \text{ coſ. } x$. Alſo $Ha = AD \times DE \div AH = \frac{ab}{d} \text{ coſ. } x$, and $HB = Ha \times \text{ſin. } x = \frac{ab}{d} \text{ coſ. } x \text{ ſin. } x$; therefore $Hd - Hb = \frac{ac}{d} \cdot \frac{\text{coſ.}^2 x}{\text{tan. } x} + \left(a - \frac{ab}{d}\right) \text{coſ. } x \text{ ſin. } x$; but $\left(a - \frac{ab}{d}\right) \text{coſ. } x \text{ ſin. } x = \frac{ac}{d} \cdot \text{coſ. } x \text{ ſin. } x = \frac{ac}{d} \cdot \frac{\text{ſin.}^2 x}{\text{tan. } x}$; hence $Hd - Hb = \frac{ac}{d} \left(\frac{\text{coſ.}^2 x + \text{ſin.}^2 x}{\text{tan. } x}\right) = \frac{ac}{d \text{ tan. } x}$. By calculating the forces Hf, Hh, in a ſimilar way, we find $Hf - Hh = \frac{mk}{f \text{ tan. } y}$, and ſince $Hd - Hb$ is $= Hf - Hh$, we have

$$\frac{ac}{d\ \text{tan.}\ x} = \frac{mh}{f\ \text{tan.}\ y} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Alſo, $GH = AH\ \text{coſ.}\ x = d\ \text{coſ.}\ x$, and $HM = PH\ \text{coſ.}\ y = f\ \text{coſ.}\ y$; therefore

$$d\ \text{coſ.}\ x + f\ \text{coſ.}\ y = g \ldots\ldots\ldots\ldots (2)$$

And from the equations 1 and 2, we may eaſily determine the angles x and y, and then the poſition of the beams is known.

To find the preſſure againſt the planes, we have $AC = \frac{AB}{\text{ſin.}\ x} = \frac{ac\ \text{coſ.}\ x}{d\,\text{ſin.}\,x} = \frac{ac}{d\ \text{tan.}\ x}$, the preſſure againſt the plane GA; and $PN = \frac{PT}{\text{ſin.}\,y} = \frac{mh}{f\ \text{tan.}\ y}$, the preſſure againſt the plane MP.

Again, $cd = cH\ \text{ſin.}\ x = \frac{ac}{d} \cdot \frac{\text{coſ.}\ x\ \text{ſin.}\ x}{\text{tan.}\ x} + a\ \text{ſin.}^2\ x$, and $ab = aH\ \text{coſ.}\ x = \frac{ab}{d}\ \text{coſ.}^2\ x$, and $cd + ab = \frac{ac}{d} \times \frac{\text{coſ.}\ x\ \text{ſin.}\ x}{\text{tan.}\ x} + a\ \text{ſin.}^2\ x + \frac{ab}{d}\ \text{coſ.}^2\ x$; but $\frac{\text{coſ.}\ x\ \text{ſin.}\ x}{\text{tan.}\ x} = \text{coſ.}^2\ x$; therefore $cd + ab = a\left(\frac{c+b}{d}\ \text{coſ.}^2\ x + \frac{d}{d}\ \text{ſin.}^2\ x\right) = a$, the preſſure of the beam AH againſt the plane GM; and by a ſimilar calculation, we find $ef + gh = m$, the preſſure of the beam PH againſt the plane GM, or the whole preſſure of the beams againſt the horizontal plane, is equal to the ſum of their weights.

THIRD SOLUTION, *by* AMICUS, *the Propoſer.*

Let GA and MP, (fig. 98, pl. 5,) be the vertical planes, AH, PH the given beams placed in the required poſition, D the centre of gravity of the beam AH, and W its weight; and Q the centre of gravity of the beam PH, and w its weight. Join DQ, and make $DL : LQ :: W : w$; then L is the common centre of gravity of the beams: and when an equilibrium takes place, this centre muſt be at the loweſt point, or at the leaſt diſtance from the horizontal plane GM; that is, if LX be perpendicular to GM, then LX muſt be a minimum. Draw DI, QK perpendicular to GM, and put $a = DH$, $b = AH$, $m = QH$, $n =$

PH, p = GM, x = ∠ GHA, and y = ∠ PHM; then DI = a ſin. x, and QK = m ſin. y. Draw QV parallel to GM, meeting LX in Y, and DI in V; then LX = QK + LY, and DV = DI − QK, and LQ : QD :: LY : DV, or LY =

$$VD \times \frac{LQ}{QD} = (ID - QK)\frac{w}{w+W} = \frac{w}{w+W} \times$$

$(a\,\text{ſin.}\,x - m\,\text{ſin.}\,y)$; therefore QK + LY = m ſin. y +

$$\frac{w}{w+W}(a\ \text{ſin.}\ x - m\ \text{ſin.}\ y) = \frac{w}{w+W}(a\ \text{ſin.}\ x) +$$

$\frac{Wm}{w+W}$ ſin. y = a minimum. Take the fluxion, then

$$\frac{wa}{w+W}\,\dot{x}\ \text{coſ.}\ x + \frac{Wm}{w+W}\,\dot{y}\ \text{coſ.}\ y = 0,$$

$$\text{or } \dot{y} = -\frac{wa\dot{x}\ \text{coſ.}\ x}{Wm\ \text{coſ.}\ y}.$$

Again, GH = AH coſ. x = b coſ. x, HM = n coſ. y, and b coſ. x + n coſ. y = p, a conſtant quantity, therefore its fluxion, or $-b\dot{x}$ ſin. x $- n\dot{y}$ ſin. y = 0, and $\dot{y} = -\frac{b\dot{x}\ \text{ſin.}\ x}{n\ \text{ſin.}\ y}$:

$$\text{Hence } \frac{wa\ \dot{x}\ \text{coſ.}\ x}{Wm\ \text{coſ.}\ y} = \frac{b\ \dot{x}\ \text{ſin.}\ x}{n\ \text{ſin.}\ y},\ \text{or } \frac{wa\ \text{coſ.}\ x}{Wm\ \text{coſ.}\ y} = \frac{b\ \text{ſin.}\ x}{n\ \text{ſin.}\ y},$$

$$\text{or } \frac{n\ \text{ſin.}\ y}{Wm\ \text{coſ.}\ y} = \frac{b\ \text{ſin.}\ x}{wa\ \text{coſ.}\ x},\ \text{or } \frac{n}{Wm}\ \text{tan.}\ y = \frac{b}{wa}\ \text{tan.}\ x,$$

and from this equation, and the equation b coſ. x + n coſ. y = p, we may determine x and y.

A ſolution was alſo received from Dr. O'Riordan.

III. QUESTION 183, *by* QUIDAM.

Given all the four ſides, and the difference of the parts into which one of the angles is divided by a diagonal; to conſtruct the trapezium and point out the limits.

SOLUTION, *by Dr.* MICHAEL STUART O'RIORDAN, *Carlow, Ireland.*

Suppoſe it done, and let ABCD, (fig. 98, pl. 5,) be the trapezium required, in which the four ſides and the difference of the angles DCA, BCA, (made by one of the diagonals AC) are given. Let the triangle AOC be made equal to the triangle ADC, and join BO; that is, let OC be made $=$ to CD, and AO $=$ to AD, and then the angle ACO will be equal to the angle ACD, and the angle BCO will be equal to the difference of the angles DCA, BCA: hence the following

CONSTRUCTION. With the given ſides CB, CO (CD), and given difference of the angles make the triangle CBO, and with the given diſtances OA (AD), BA, and centres O, B, deſcribe arcs to interſect at A; make the triangle ACD $=$ to AOC, and it is done.

It is evident, that BO muſt not be leſs than the difference, nor greater than the ſum of OA, BA; and when BO is equal to the difference of AO, BA the angle BCO, or the difference of the angles DCA, BCA is a minimum.

Meſſrs. Amicus *and* Cunliffe, *anſwered it.*

IV. QUESTION 184, *by Mr.* JOHNSON, *Birmingham.*

A given quantity of gold is to be made into a cup in the form of a ſegment of a ſphere, and n tenths of an inch in thickneſs: what are the dimenſions of the cup when it is made ſo as to contain the greateſt quantity poſſible?

SOLUTION, *by Mr.* CUNLIFFE.

Let r denote the radius of the interior ſphere, or that which conſtitutes the hollow part of the cup, and let x denote the depth thereof, or the verſed ſine of the ſegment: then will $r+n$ denote the radius of the exterior ſphere, and $x+n$ will denote the verſed ſine of its ſegment. And by a well known theorem, $p \times (rx^2 - \frac{1}{3}x^3)$, will expreſs the content of the hollow part, where $p = 3{\cdot}1416$; and $p \times \left\{(r+n)(n+x)^2 - \frac{1}{3} \times (n+x)^3\right\}$ will expreſs the content of the ſegment of the exterior ſphere. Alſo the difference of the foregoing expreſſions, viz. $p \times \left\{(r+n)(n+x)^2 - rx^2\right\} - \frac{1}{3}p \times (n+x)^3 + \frac{1}{3}px^3$ will expreſs the quantity of gold in the cup, which put $= pq^2n$.

Now the content of the hollow part, viz. $p \times (rx^2 - \frac{1}{3}x^3)$ is to be a maximum by the queſtion, or $rx^2 - \frac{1}{3}x^3$ is to be a maximum; putting its fluxion $= 0$, gives $\dot{r}x^2 + 2rx\dot{x} - x^2\dot{x} = 0$; whence $\dot{r} = -\dot{x}(2r - x) \div x$.

Furthermore, the expreſſion for the quantity of metal being conſtant, its fluxion will be $= 0$, or $\dot{r}(n+x)^2 + 2\dot{x}(r+n)(n+x) - \dot{r}x^2 - 2rx\dot{x} - \dot{x}(n+x)^2 + x^2\dot{x} = \dot{r}(n^2+2nx) + \dot{x}(n^2+2rn) = 0$, or $\dot{r} = -\dot{x} \times \frac{n+2r}{n+2x}$; therefore $\frac{n+2r}{n+2x} = \frac{2r-x}{x}$ which equation, when properly reduced, gives $x = r$. And hence it appears, that the required ſegment is an hemiſphere. Having now determined that the required ſegment is an hemiſphere, it will be eaſy to find the radius of the ſphere; for, by a well known theorem,

$\frac{2}{3}p \times \{(r+n)^3 - r^3\} = \frac{2}{3}p \times (3r^2n + 3rn^2 + n^3) = pq^2n$, by the queſtion: whence, after proper reduction, completing the ſquare, &c. we ſhall have $r = \frac{1}{6}\sqrt{(18q^2 - 3n^2)} - \frac{1}{2}n$.

This queſtion was anſwered by Meſſrs. O'Riordan *and* Toplis.

V. QUESTION 185, *by* QUIDAM.

Let one end of a ſtring of a given length be faſtened to a point A in the circumference of a circle, and laid tightly upon the circumference, and brought with the other end T to meet the diameter AB in T: what muſt be the diameter of the circle that the ſpace BTP included by the right lines TP, TB, and the arc BP may be the greateſt poſſible, TP being a tangent at T?

SOLUTION, *by Mr.* CUNLIFFE.

Draw the radius CP, (fig. 99, pl. 5,) then PT $\times \frac{1}{2}$CP $=$ the area of the triangle TPC, and arc BP $\times \frac{1}{2}$CP $=$ area of the ſector BPC, and their difference, *viz.* (PT — arc BP) $\times \frac{1}{2}$CP $=$ the ſpace BTP included by the right lines TP, TB and the arc BP.

Now let q denote the quadrantal arc of a circle, whoſe radius is 1; and let a denote the length of the ſtring APT; then arc AP + PT $= 2q \times$ CP — arc BP + PT $= a$, whence PT — arc BP $= a - 2q \times$ CP, and hence the ſpace BTP $=$ (PT — arc BP) $\times \frac{1}{2}$CP $= (a - 2q \times \text{CP}) \times \frac{1}{2}\text{CP} = q \times (a \div 2q - \text{CP}) \times \text{CP} =$ a maximum, or $(a \div 2q - \text{CP}) \times \text{CP}$ a maximum.

The problem is therefore now evidently reduced to the dividing of a line of a given length into two ſuch parts, that their rectangle may be a maximum, which is known to be the caſe when the given line is bisected; hence we shall have $CP = a \div 4q$, or $AB = 2CP = a \div 2q$. *Q. E. I.*

From the preceding conclusions, it is easily perceived, that having given the length of the string APT, we may determine the radius CP, so that the area BPT may be equal to a given space, by means of a quadratic equation.

Or which amounts to the same thing, the solution of the following question may be effected by a quadratic equation.

Let a string, TPAP′, of a given length, having its ends fastened together, be laid upon the periphery of a circle, and stretched to point T, in the plane of the circle; to determine the radius so that the area PTP′, included by the tangents TP, TP′, and the intercepted arc PP′ may be equal to a given space.

An ingenious solution was received from Dr. O'Riordan.

VI. QUESTION 186, *by Mr.* JOHN WHITLEY.

If an ellipſe be circumſcribed by a triangle, and lines be drawn from the angles of the triangle to the points of contact, meeting the ellipſe again in three points; tangents drawn to the curve at theſe points will interſect each other in the ſaid lines produced. Alſo, if the ſaid tangents and the ſides of the triangle be produced till they meet, the three points of interſection will be in the ſame ſtraight line. Required the demonſtration?

SOLUTION, *by Mr.* LOWRY, *R. M. College.*

The demonstration of this proposition will be greatly simplified by premising the following Lemma.

Let three straight lines, AB, BC, AC, (fig. 100, pl. 5,) touch a conic section in the points P, Q, R, and meet each other in A, B, C; then, if straight lines AQ, BR, CP be drawn from the points A, B, C, to the points of contact Q, R, P, meeting the curve in a, b, c, they will intersect each other in the same point K,

Let the line which joins the points Q, R, meet AB in Z, and CP in N, and let K be the intersection of the lines AQ, BR; then if PC does not pass through K, let PK be drawn to meet QR in some point n different from N; then (Hamilton's Conic Sections, Prop. IV. B. V.) the line AZ is harmonically divided in the points A, P, B, Z; therefore the lines AC, PC, BC, are harmonicals with respect to the point Z; and the lines AQ, Pn, BR, drawn through K, are also harmonicals with respect to the said point Z: wherefore the line ZR, which meets both sets of harmonicals, is harmonically divided in the points Q, N, R, Z, and also in the points Q, n, R, Z, which is impossible, except the points N and n coincide; therefore PC must pass through K.

When KQ is parallel to AB, it is bisected in N, and AB is then bisected in P, and the truth of the lemma is evident.

Cor. A tangent to the curve at *c*, passes through the point Z; and the points *a*, *b*, Z are in a straight line.

For if Z*c* is not a tangent to the curve, let Z*d* touch the curve at *d*, and draw P*d* meeting RQ in *m*; then RZ is harmonically divided in the points R, *m*, Q, Z (Ham. P. III. B. V.) and also in the points R, N, Q, Z (as is proved above,) which is impossible; therefore *m* coincides with N, and *d* with *c*, and Z*c* is a tangent at *c*.

Again, if the points *a*, *b*, Z, are not in a straight line, let Z*b* be produced to meet the curve in M, and the lines CP, AQ in I and L; then because of the harmonicals, LZ is harmonically divided in the points L, I, *b*, Z; and (Ham. P. III. B. V.) it is also harmonically divided in the points M, I, *b*, Z; which is impossible, except L and M coincide with the point *a*, (Ham. B. V. Def. 2, and Cors.); therefore the points *a*, *b*, Z, are in a straight line.

Hence it appears, that the lines RQ, *ab*, and the tangents to the curve at P and *c*, intersect each other in the same point Z.

Demonstration of the Proposition. Fig. 101, pl. 5.

Let the figure be drawn as before, and let the tangents at the points *a*, *b*, *c*, intersect in E, F, and G. Then, 1st. the proposition affirms that the intersections E, F, G, are in the lines AQ, BR, CP produced. Now, (Ham. P. VIII. B. V.) the intersection A of the tangents RA, PA; the intersection E of the tangents *b*E, *c*E; and the intersection K, of the lines R*b*, *c*P, are in a straight line; but it has been shewn in the lemma, that the intersection K, is in the line AQ, and therefore the intersection E is in the line AQ produced. In the very same way it may be shewn, that the point F is in the line BR produced, and that the point G is in the line CP produced.

It is manifest, (Ham. ibid.) that the lines R*c*, P*b*, will intersect in AE produced: that the lines P*a*, Q*c*, will intersect in BR produced; and finally, that the lines R*a*, Q*b*, will intersect in GC produced.

2dly. Let the tangents at *a*, *b* and *c*, (fig. 102, pl. 5,) meet the tangents at Q, R, and P, in X, Y, and Z; then, by the proposition last cited, the intersection X of the tangents *a*X, QX; the intersection Y of the tangents *b*Y, RY; and the intersection of the lines RQ, *ab*, are in a straight line. But it has been proved, in the cor. to the lemma, that the lines RQ, *ab*, intersect in the point Z, where the tangents PZ, *c*Z, intersect, and therefore, the points X, Y, Z, are in a straight line.

The last part of the question is only a particular case of the general proposition enunciated at page 144 of the last number of the Repository, when applied to the conic sections.

Dr. O'Riordan *also answered this question.*

VII. QUESTION 187, *by* LAPUTIENSIS.

Let two ſpherical balls given in magnitude and weight, and having their ſurfaces perfectly poliſhed, be put into a ſpherical veſſel of a given diameter, whoſe internal ſurface is alſo perfectly poliſhed: required the diſtances of the points of contact from the loweſt point of the veſſel, when the balls reſt in equilibrio?

SOLUTION, *by the Rev.* I. TOPLIS.

Let OEF*e* (fig. 103, pl. 5,) be a section of the vessel and spherical balls, perpendicular to the horizon, and passing through their centres; O the centre of the vessel, OF a vertical line, OAE, and O*ae*, lines passing through the centres of the balls to the vessel, AM*a* a line joining their centres. Let AD perpendicular to the horizon, represent the weight of the ball AEM, and DE be drawn parallel, and AC meeting it, perpendicular to OAE; then the force AD may be supposed to be decomposed into the forces AC, which accelerates the sphere down the side of the vessel, and CD, which acts against the surface of the vessel at E. As A*a* joins the centres of the spheres, it passes through their point of contact M, and the force which opposes the descent of the sphere AEM, acts in the direction MA: let BA represent this force which may be decomposed into two others, one parallel to OAE, and the other perpendicular to it; the force parallel to OE, is opposed by the surface EF, and that perpendicular to it, ought to be equal to AC, in the case of an equilibrium; therefore these forces may be represented by BC and CA. If in the sphere M*ea*, *ad* is taken perpendicular to the horizon, and in the same ratio to AD, as the weight of the ball M*ea* is to that of MEA, and *deb* is drawn parallel, and *ac* perpendicular to O*e*, the force represented by *ad*, may be decomposed into *cd*, which is opposed by the re-action of the vessel, and *ac* which accelerates the sphere down the vessel. As the force which opposes the rolling of the ball, acts in the direction M*a*, let it be represented by *ba*, which, in the case of an equilibrium, is equal to BA; then *ba*, when decomposed into two forces, one parallel, and the other perpendicular to O*ae*, that perpendicular to it, must evidently be represented by *ca*. As the lines OE, AE, and *ae* are given, their sums and differences are given, therefore OA, O*a*, and A*a*, and consequently the triangle OA*a* are given.

Put A*a* $= a$, AD $= W$, *ad* $= w$, and $x =$ AG; then *a*G $= a - x$, and, by similar triangles, OG : AG :: AD : AB $= xW \div \text{OG}$; also, OG : *a*G :: *ad* : *ab* $= (a-x)w \div \text{OG}$, and since AB $=$ *ab*, we have $xW \div \text{OG} = (a-x)w \div \text{OG}$, or $xW = (a - x)\,w$; therefore $x = a \div (W + w)$. And as in the triangle OAG, AO, AG and ∠OAG are known the remaining sides and angles may be easily found, and in a similar manner the sides and angles of the triangle OG*a* are known: but the angles EOF and FO*e* give the distances of the points of contact from the lowest point F. Join FM and in the triangle FGM we know MG = AM — AG, GF = OF — OG and ∠MGF = ∠AGO, from which we may find MF, or the distance of M from the lowest point.

SECOND SOLUTION, *by Dr.* O'RIORDAN.

Let OEF*e*, (fig. 103, pl. 5,) represent a vertical section of the balls and vessel; A and *a* the centres of the balls; E and *e* the points of contact with the internal superficies of the vessel: draw A*a*, which will manifestly be equal to the sum of the semidiameters of the two balls, and therefore given; also produce the radii EA, *ea*, till they meet, which will be in O, the centre of the vessel. Now the common centre of gravity of the two balls, will be a given point in A*a*, which suppose G; then through O and G, draw the right line OGF = the radius EO, and join EF, *e*F.

When the balls are in equilibrio, the line OGF must be perpendicular to the horizon, and F will be the lowest point of the vessel, and EF, *e*F, the required distances of the points of contact of the balls from the lowest point.

The calculation will be very easy: AO*a* is a given triangle, and G a given point in the base A*a*; whence we find ∠ AOG = ∠FOE; there is also given EO = FO, whence we may easily find EF. And in the same manner *e*F may be found.

Solutions were alſo received from Messrs. Amicus *and* Cunliffe.

VIII. QUESTION 188, *by Mr.* IVORY.

Two points being given in the ſame vertical plane, but not in one horizontal line; it is required to determine the poſition of two inclined planes ſuch, that the time of deſcent from the higher point to the lower, on theſe two planes, ſhall be leſs than on any other two inclined planes whatſoever?

SOLUTION, *by Mr.* IVORY, *the Proposer.*

Let A and B, (fig. 104, pl. 5,) be the two points, and AG, GB, the two inclined planes required: draw the horizontal lines AF and GE, and the vertical lines GD and BF: draw also AH perpendicular to BG.

Let $g = 32\frac{1}{6}$ feet, the accelerating power of gravity: Then the time of descent on the inclined plane, AG will be = $(AG \div GD) \times \sqrt{(BD \div \tfrac{1}{2}g)} = \tau$:

The velocity acquired by the fall $= \sqrt{(2g \times DG)}$:

The velocity, in the direction of the plane GB, $= (GH \div GA) \times \sqrt{(2g \times DG)} = V$.

Let τ' be the time of descent on the inclined plane GB, then $V \times \tau' + (BE \div BG) \times \tfrac{1}{2} g \tau'^2 = BG$: hence

$$2 \frac{GH}{GA} \cdot \frac{GB}{BE} \cdot \sqrt{DG} \times (\tau' \sqrt{\tfrac{1}{2}g}) + (\tau'^2 \sqrt{\tfrac{1}{2}g})^2 = \frac{BG^2}{BE^2} \cdot BE\text{: and}$$

$$\tau' \sqrt{\tfrac{1}{2}g} = \frac{BG}{BE} \left\{ \sqrt{\left(BE + \frac{GH^2}{GA^2} \cdot DG\right)} - \frac{GH}{GA} \sqrt{DG} \right\}. \text{ Therefore}$$

$$(\tau + \tau') \sqrt{\tfrac{1}{2}g} = \frac{BG}{BE} \left\{ \sqrt{\left(BE + \frac{GH^2}{GA^2} \cdot DG\right)} - \frac{GH}{GA} \cdot \sqrt{DG} \right\} + \frac{AG}{GD} \sqrt{GD}.$$

If we put $AF = b$, $BF = a$, $GD = x$, and $AD = y$, the expression for $(\tau + \tau') \sqrt{\tfrac{1}{2}g}$, which is to be a minimum, will become a function of x and y, for which we may write $f(x, y)$: then the equations for determining x and y, in the case of a minimum, will be had by putting the partial fluxions of $f(x, y)$ (the variables being x and y) separately $= 0$: that is the equations will be $\left(\frac{df(x, y)}{dx}\right) = 0$, and $\left(\frac{df(x, y)}{dy}\right) = 0$.

But as the preceding expression for the whole time of the fall seems to be somewhat complicated, in order to simplify it, we shall suppose that the falling body passes from the upper plane to the lower, without any loss of velocity, as in the case of a continued curvature; for this purpose, we have only to write 1 in place of $GH \div GA$ in the preceding expression, and we obtain $f(x, y) = (BG \div BE)(\sqrt{BF} - \sqrt{GD}) + (BA \div GD)\sqrt{GD}$:

And if we put $\varrho = BA = \sqrt{(x^2 + y^2)}$, $\varrho' = BG = \sqrt{\{(b - y)^2 + (a - x)^2\}}$, then $f(x, y) = \frac{\varrho'}{a - x}(\sqrt{a} - \sqrt{x}) + \frac{\varrho}{x}\sqrt{x}$:

And remarking that $\left(\frac{d\left(\frac{\varrho'}{a - x}\right)}{dy}\right) = -\frac{b - y}{\varrho'} \cdot \frac{1}{a - x}$; $\left(\frac{d\left(\frac{\varrho'}{a - x}\right)}{dx}\right) = \frac{(b - y)^2}{(a - x)^2} \cdot \frac{1}{\varrho'}$; $\left(\frac{d\left(\frac{}{x}\right)}{dy}\right) = \frac{y}{\varrho} \cdot \frac{1}{x}$; $\left(\frac{d\left(\frac{\varrho}{x}\right)}{dx}\right) = -\frac{y^2}{x^2} \times \frac{1}{\varrho}$;

we shall have according to the method noticed above

$$\left(\frac{df(x, y)}{dy}\right) = -\frac{b - y}{\varrho} \cdot \frac{\sqrt{a} - \sqrt{x}}{a - x} + \frac{y}{\varrho} \cdot \frac{\sqrt{x}}{x} = 0,$$

$$\left(\frac{df(x, y)}{dx}\right) = \frac{(b - y)^2}{(a - x)^2} \cdot \frac{\sqrt{a} - \sqrt{x}}{\varrho'} - \frac{y^2}{x^2} \cdot \frac{\sqrt{x}}{\varrho} - \frac{1}{2}\left(\frac{\varrho'}{a - x} - \frac{\varrho}{x}\right) \cdot \frac{1}{\sqrt{x}} = 0$$

whence $\frac{b-y}{\varrho'} \cdot \frac{\sqrt{x}}{\sqrt{a}+\sqrt{x}} = \frac{y}{\varrho}$

$$\frac{b-y}{a-x} \times \frac{b-y}{\varrho'} \times \frac{\sqrt{x}}{\sqrt{a}+\sqrt{x}} - \frac{y}{x} \times \frac{y}{\varrho} - \tfrac{1}{2}\left(\frac{\varrho'}{a-x} - \frac{\varrho}{x}\right) = 0.$$

Let φ and ψ denote the angles which the planes AG and GB, make with the horizon; then $\frac{b-y}{a-x} = \frac{\cos\psi}{\sin\psi}$, $\frac{y}{x} = \frac{\cos\varphi}{\sin\varphi}$,

$$\frac{\varrho'}{a-x} = \frac{1}{\sin\psi},\ \frac{\varrho}{x} = \frac{1}{\sin\varphi},\ \frac{b-y}{\varrho'} = \cos\psi,\ \frac{y}{\varrho} = \cos\varphi.$$

we have $\cos\psi \times \frac{\sqrt{x}}{\sqrt{a}+\sqrt{x}} = \cos\varphi$,

$$\frac{\cos^2\psi}{\sin\psi} \times \frac{\sqrt{x}}{\sqrt{a}+\sqrt{x}} - \frac{\cos^2\varphi}{\sin\varphi} = \tfrac{1}{2}\left(\frac{1}{\sin\psi} - \frac{1}{\sin\varphi}\right).$$

And if we exterminate $\sqrt{x} \div (\sqrt{a}+\sqrt{x})$, we shall have $2\cos\varphi \times \sin(\varphi-\psi) = \sin\varphi - \sin\psi$: this equation seems to require $\varphi = \psi$, which being inconsistent with the equation, $\cos\psi \times \sqrt{x} \div (\sqrt{a}+\sqrt{x}) = \cos\varphi$, will give no solution: but remarking that $\sin\psi = \sin(\varphi-(\varphi-\psi)) = \sin\varphi\cos(\varphi-\psi) - \cos\varphi\sin(\varphi-\psi)$, we get by substitution, $\cos\varphi\sin(\varphi-\psi) = \sin\varphi(1-\cos(\varphi-\psi))$; whence $\frac{\sin\varphi}{\cos\varphi} = \tan\varphi = \frac{\sin(\varphi-\psi)}{1-\cos(\varphi-\psi)} = \tan\left(90^\circ - \frac{\varphi-\psi}{2}\right)$: therefore $3\varphi - \psi = 180^\circ$, and $\cos\psi = -\cos 3\varphi$, $\sin\psi = -\sin 3\psi$: or $\cos\psi = 3\cos\varphi - 4\cos^3\varphi$, and $\sin\psi = 4\sin^3\varphi - 3\sin\varphi$: Hence, $\frac{\cos\psi}{\cos\varphi} = 3 - 4\cos^2\varphi = 1 - 2\cos 2\varphi = \frac{\sqrt{a}+\sqrt{x}}{\sqrt{x}}$

put $z = \cos 2\varphi$, then $-2z = \sqrt{(a \div x)}$, and $x = a \div 4z^2$.

Again, $AD = \frac{x\cos\varphi}{\sin\varphi}$, and $GE = \frac{(a-x)\cos\psi}{\sin\psi} = \frac{\cos\varphi}{\sin\varphi} \times \frac{(a-x)(3-4\cos^2\varphi)}{4\sin^2\varphi-3}$: therefore

$\frac{\cos\varphi}{\sin\varphi} \times \left(x + \frac{(a-x)(3-4\cos^2\varphi)}{4\sin^2\varphi-3}\right) = b$: And because $x = \frac{a}{4z^2}$; $\frac{\sin\varphi}{\cos\varphi} = \sqrt{\frac{1-z}{1+z}}$; $2\cos^2\varphi = 1+z$, $2\sin^2\varphi$

=

$= 1 - z$, we get by substituting and reducing

$$(1 - 2z + 2z^2)^2\ (1 + z) = \frac{b^2}{a^2}(4z^4 - 4z^5)$$

whence the problem may be resolved by finding the value of z, or cos 2φ, in an equation of the 5th degree.

IX. QUESTION 189, *by Mr.* CUNLIFFE.

Required the ſum of the infinite ſeries

$$\tfrac{1}{2} - \tfrac{1}{3} \times (\tfrac{1}{2}) + \tfrac{1}{4} \times (\tfrac{1}{2} + \tfrac{1}{3}) - \tfrac{1}{5} \times (\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}) + \tfrac{1}{6} \times (\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5}) - \&c?$$

SOLUTION, *by Dr.* O'RIORDAN.

Put $z = \frac{x^2}{2} - (\frac{1}{2}) . \frac{x^3}{3} + (\frac{1}{2} + \frac{1}{3}) . \frac{x^4}{4} - (\frac{1}{2} + \frac{1}{3} + \frac{1}{4}) . \frac{x^5}{5} + \&c.$

Take the fluxions and divide by $\dot{x}$, then

$$\frac{\dot{z}}{\dot{x}} = x - \frac{x^2}{2} + (\tfrac{1}{2} + \tfrac{1}{3}) . x^3 - (\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}) . x^4 + (\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5}) . x^5 - \&c.$$

Multiply by $1 + x$, and take x^2 from both sides of the equation, then $\frac{\dot{z}}{\dot{x}}\ (1 + x) - x^2 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.$

Let φ be the hyp. log. of $1 + x$, then $\varphi = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.$ and $\dot{\varphi} = \frac{\dot{x}}{1 + x}$, and by substitution, $\frac{\dot{z}}{\dot{\varphi}} - x^2 = \varphi$, or $\dot{z} = \varphi\dot{\varphi} + x^2\,\dot{\varphi}$

And taking the fluents, $z = \frac{1}{2}\varphi^2 + \varphi - x + \frac{1}{2}x^2$, which needs no correction. Let x be taken $= 1$, then φ is the log. of 2, and $z = \frac{1}{2}\varphi^2 + \varphi - \frac{1}{2}$ the sum required.

Mr. Cunliffe, the proposer, by assuming $\int \frac{\dot{x}}{1 + x} \times (x - ax^2 + bx^3 - cx^4 + \&c.) = \frac{1}{2}x^2 - \frac{1}{3}Px^3 + \frac{1}{4}Qx^4 - \frac{1}{5}Rx^5 + \&c.$ and proceeding as in his solution to question 160, (pa. 87 of the present vol.) obtains the same result as above.

The sum of the series may also be immediately found from Mr. Ivory's solution on page 117 of the present vol.; for if a be taken $= 1$, then $\frac{1}{2}$ the square of the log. of 2 is $= \frac{1}{2} - (1 + \frac{1}{2}) . \frac{1}{3} + (1 + \frac{1}{2} + \frac{1}{3}) . \frac{1}{4} - (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}) . \frac{1}{5} + \&c. =$ to the series in the question, together with the series $- \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \&c.$ But this last series is evidently $=$ to $\frac{1}{2} -$ log. 2, and therefore the sum of the given series is $= \frac{1}{2}\log.^2 2 - (\frac{1}{2} - \log. 2,)$ as before.

The next four Queſtions are ſelected from the Cambridge Univerſity Calendar for 1806.

X. QUESTION 190, *from the* CAMB. UN. CAL.

A given cylindric veſſel is ſupplied with water by a cock at a given rate. Then ſuppoſe, when the veſſel is full, that an aperture of a given area is made in its bottom; what is the loweſt point to which the ſurface of the water in the veſſel will deſcend, and alſo the time of its deſcent; the influx of the water by the cock being ſuppoſed ſlower than the efflux when the veſſel is full?

SOLUTION, *by Mr.* CONLIFFE.

Let h denote the height, and d the diameter of the vessel; A the area of the aperture or hole made in the bottom. Also let x denote any variable height of the water in the vessel whilst running, $g = 16\frac{1}{12}$; and suppose the velocity of the effluent water to be the same as would be generated by gravity in a heavy body falling freely from the height of the surface of the water above the aperture in the bottom, and put $p = \cdot 7854$. Then $2\sqrt{gx}$ will express the velocity of the effluent water per second, when the surface is at the height x above the hole in the bottom: and, therefore, $2Agx$ will express the quantity of water issuing per second, when at the height x. Let $2An$ denote the quantity of water per second, supplied by the cock, and then $2A\sqrt{gx} - 2An = 2A \times (\sqrt{gx} - n)$ will express the rate of evacuation of the vessel per second, when the water is at the height x above the aperture; and therefore $\frac{-d^2 y\dot{x}}{2A(\sqrt{gx} - n)}$ will express the fluxion of the time of evacuation. The fluent of this expreſſion, viz.

$$\frac{nd^2 p}{Ag} \times \text{h. l.} (\sqrt{gx} - n) - \frac{d^2 p\sqrt{gx}}{Ag},$$

expresses the time of evacuating to the height x above the hole But this expression ought to vanish when $x = h$, therefore the correct expression for the time of evacuating to the height x, will be

$$\frac{d^2 p}{Ag} \times (\sqrt{gh} - \sqrt{gx}) - \frac{nd^2 p}{Ag} \times \text{h. l.} \frac{\sqrt{gh} - n}{\sqrt{gx} - n}.$$

Again it is evident, that the surface of the water in the vessel cannot sink below the point where the rates of the influx and efflux per second, are equal to each other; that is, where $2A . \sqrt{(gx - n)} = 0$, or where $x = n^2 \div g$, in which circumstance the preceding correct expression for the time becomes

$$\frac{d^2 p}{Ag} \times (\sqrt{gh} - n) + \frac{nd^2 p}{Ag} \times \text{h. l.} \frac{\sqrt{gh} - n}{0},$$

which is infinite.

Whence it appears, that the surface of the water in the vessel can never arrive at the point, where the rates of influx and efflux are equal to each other.

Dr. O'Riordan *answered this question.*

XI. QUESTION 191, *from the* CAMB. UN. CAL.

To this question we have received no answer.

XII. QUESTION 192, *from the* CAMB. UN. CAL.

A perfectly flexible chain is wound round a cylinder ſupported with its axis parallel to the horizon. Then, if the weight and dimenſions of the cylinder be given, and alſo the length and weight of the chain; it is required to determine the time in which the chain, impelled by the force of gravity, will unwind itſelf; a given length being unwound at the commencement of the motion?

SOLUTION, *by Mr.* CUNLIFFE.

The solution of the following problem will greatly facilitate the answer to the question.

Suppose a cylinder of uniform matter of given dimensions and weight, to be at liberty to turn freely about its axis: it is required to determine what weight must be placed at a point upon the surface of an hollow cylinder of equal dimensions, so as equally to resist the communication of motion by any equal forces acting at the surfaces of the two cylinders, in directions at right angles to the radii.

Let SRC (fig. 105, pl. 5,) represent a section of the cylinder at right angles to its axis, SC a radius thereof; S*a*A a circle concentrical to SRC. Put SC $= r$, SA $= z$, and $c = 3{\cdot}1416$; and let 2B denote the weight of the cylinder. Then $2cz =$ the circumference of the circle S*a*A, and $2cz\dot{z} =$ the fluxion of the space S*a*A. And we shall manifestly have $r^2c : 2cz\dot{z} :: 2B :$

$4z\dot{z}B \div r^2 =$ the fluxion of the weight of the circle or cylinder S*a*A, and the last expression being written for A in the expression, art. 395, page 398, vol. ii, last line but two, Dalby's Course; the expression there becomes $4z^3\dot{z}\,B \div r^4$, which is the fluxion of the weight to be placed at C: taking the fluents, we have $z^4B \div r^4$, which when $z = r$ becomes B. Hence it appears, that half the given weight of the cylinder being placed at a point in the surface, would equally resist the communication of motion by any force acting at a point C, in a direction CP, at right angles to SC, as the cylinder itself would.

Solution of the question.

Let 2B denote the weight of the cylinder, l the length of the chain, and w its weight; and let a denote the length of the part of the chain depending from the cylinder at the commencement of the motion, and let $CP = x$, denote a variable part of the chain depending from the cylinder whilst in motion, and let v denote the corresponding velocity of the end P.

Then we shall obviously have $l : x :: w : wx \div l =$ the weight of the part x, and therefore $2gwx \div l$ expresses the motive force urging the system, when the length x of the chain hangs from the cylinder. But it appears from the preceding problem, that $B + w$ expresses the whole inertia of the system; therefore $2gwx \div l\,(B + w)$ will express the accelerative force, when the length x of the chain depends from the cylinder. Then, by a well known theorem, $\frac{2gwx\dot{x}}{l\,(B + w)} = v\dot{v}$; taking the correct fluents, by making the resulting expression to vanish when $x = a$, we shall have $v^2 = \frac{2gw\,(x^2 - a^2)}{l\,(B + w)}$.

Again, $\dot{t} = \frac{\dot{x}}{v} = \sqrt{(\frac{l\,(B + w)}{2gw})} \cdot \frac{\dot{x}}{\sqrt{(x^2 - a^2)}} =$ the fluxion of the time of unwinding to the length x. And the correct fluents being taken, gives $t = \sqrt{(\frac{l\,(B + w)}{2gw})} \times \text{h.l.}\,\frac{x + \sqrt{(x^2 - a^2)}}{a}$ where writing l for x, it becomes $\sqrt{(\frac{l\,(B + w)}{2gw})} \times \text{h.l.}\,\frac{l + \sqrt{(l^2 - a^2)}}{a}$ which expresses the time of unwinding the chain.

A solution was also received from Dr. O'Riordan.

XIII.

XIII. QUESTION 193, *from the* CAMB. UN. CAL.

A cylindrical veſſel, whoſe height is equal to its diameter, is filled with water; with what velocity muſt it be whirled round its axis, that half the water may be thrown out?

SOLUTION, *by Mr.* CUNLIFFE.

I shall first solve the following problem; by means of which, an answer to the question will be readily obtained.

To find the nature of the curve, which the surface of a fluid contained in a cylindrical vessel acquires, whilst the vessel revolves about its axis with an uniform velocity.

Let ABCD, (fig. 106, pl. 5,) represent a section of the cylindrical vessel, by a plane passing through the axis EF; AVB a section of the curved surface which the fluid acquires by the circular motion about EF. Draw the perpendicular ordinate HG, and through the vertex V, draw VR parallel thereto, meeting HR parallel to EF in R.

Put EB $= r$, $c =$ the angular velocity per second, at the distance r, from the axis EF; VR $=$ GH $= y$, VG $=$ RH $= x$, and $g = 16\frac{1}{12}$; then $r : y :: c : cy \div r =$ the angular velocity at R: and hence $c^2y \div r^2 =$ the centrifugal force urging a particle of the fluid at R, and $c^2y\dot{y} \div r^2 =$ the fluxion of the aggregate of the forces of all the particles in the line VR; the fluent whereof is $c^2y^2 \div 2r^2$, which, from the principles of hydrostatics, must be equal to the pressure of all the particles in the line HR; that is, $c^2y^2 \div 2r^2 = 2gx$, or $4r^2gx \div c^2 = y^2$, an equation expressing the nature of the curve AVHB; and which, from thence appears to be the common parabola.

Solution of the question.

It has just been proved, that the hollow part of the cylinder of water, caused by the whirling motion round the axis, is in the form of a conic paraboloid; and the content of a conic paraboloid is known to be one half of its circumscribing cylinder.

Moreover the equation of the curve was found to be $4r^2gx \div c^2 = y^2$, the quantities x, y, &c. being as specified, and when x becomes $= 2r$, and $y = r$, then $c^2 = 8rg$, or $c = 2\sqrt{(2rg)}$. That is, the velocity per second, with which a point of the cylindrical surface must be whirled about the axis, in the specified case, must be equal to that which an heavy body would acquire, by descending freely from rest by the force of gravity, the whole height of the cylinder. *Q. E. I.*

Dr. O'Riordan *answered it.*

XIV, QUESTION 194, *by* AMICUS.

Of all the ellipſes that can be inſcribed in the ſame given quadrilateral, required that which approaches the neareſt to a circle?

SOLUTION, *by Mr.* LOWRY.

Let ACES, (fig. 107, pl. 5,) be the given quadrilateral, B*abd* the inscribed ellipse, O its centre, BOH a diameter drawn from the point of contact B, and OL a semi-diameter parallel to BA, and which will be conjugate to BH. Let p be the semi-transverse, and q the semi-conjugate axis, then when the ellipse approaches the nearest to a circle, $p \div q$ must be a minimum.
Now $p^2 + q^2 = BO^2 + OL^2 = m^2$, (Emerson on the Ellipse, p. 34.
and $pq = BO . OL . \sin LOB = n^2$ (Ibid. p. 37.)
or putting $p \div q = r$, we have $p^2 = r^2q^2$;
and $r^2q^2 + q^2 = m^2$, or $q^2 = m^2 \div (r^2 + 1)$,
and $rq^2 = n^2$, or $q^2 = n^2 \div r$;
therefore, $m^2 \div (r^2 + 1) = n^2 \div r$, or $rm^2 = n^2r^2 + n^2$,

and $r = \frac{p}{q} = \frac{m^2}{2n^2} + \sqrt{(\frac{m^4}{2n^4} - 1)}$, which is to be a minimum.

Produce AS, CE, till they meet in V, and draw IF to touch the ellipse at H, the extremity of the diameter BH, meeting AS in I, DS in G, and CV in F; then IF is parallel to AC.

Put AC $= a$, AD $= b$, AV $= c$, AS $= d$, SV $= e$, AB $= x$, and IV $= y$; then AB.IH $=$ OL^2 $=$ BC.HF, ibid. p. 47; or AB : BC :: HF : IH, and by composition, AC : BC :: IF : IH $=$ IF $\times (a - x) \div a$. Also, AB.IH $=$ OL^2 $=$ BD.IG, or AB : BD :: IG : IH; and, by composition, AD : BD :: IG : IH $=$ IG $\times (b - x) \div b$; therefore, IF $\times (a - x) \div a =$ IG $\times (b - x) \div b$. But by similar triangles,
AV : AC :: IV : IF $= ay \div c$,
and AS : AD :: IS : IG $= b(y - e) \div d$;

therefore $(a - x)\frac{y}{c} = (b - x)(\frac{y - e}{d})$,

Draw IW parallel to BH, then IH being $= (a - x)y \div c$, AW is $=$ AB $-$ IH $= x - (a - x)y \div c$.

Put $s =$ the sine, and $t =$ the cosine of the given angle IAW, then by trigonometry, IW^2 ($=$ BH^2) $=$ AI^2 $+$ AW^2 $- 2t$. AI.AW; but AI$^2 = (c - y)^2$, AW$^2 = \{x \div (a - x)y \div c\}^2$, $2t$.AI.IW $= 2t(c - y)\{x - (a - x)y \div c\}$, 4OL^2 $=$ 4AB $\times$ IH $= 4x(a - x)y \div c$; and collecting the terms, we have

$$BH^2 + 4OL^2 = 4m^2 = (c - y)^2 + \{x + (a - x)y \div c\}^2 - 2t(c - y)\{x - (a - x)y \div c)\}.$$

Again, the sine of the angle AWI, or of BOL, is $= s\,(c-y) \div \mathrm{IW} = \frac{1}{2}s\,(c-y) \div \mathrm{OB}$; therefore $\mathrm{BO.OL.}\sin\angle\,\mathrm{BOL} = \frac{1}{2}s\,(c-y)\,\surd\left\{(a-x)\,yx \div c\right\} = n^2$: these values of m^2 and n^2 being substituted in the equation $\frac{p}{q} = \frac{m^2}{2n^2} + \surd\left(\frac{m^4}{2n^4} - 1\right)$, we have an expression for $p \div q$ in terms of x, y, and known quantities. By making the fluxion of this expression $=$ to 0, we obtain a value of $\dot{x} \div \dot{y}$; and by taking the fluxion of the equation $(a-x)\,y \div c = (b-x)\,(y-e) \div d$, we get another value of $\dot{x} \div \dot{y}$, which being equated with the former value, we obtain an equation expressing the relation of x and y, and eliminating x or y, by means of the equation $(a-x)\,y \div c = (b-x)\,(y-e) \div d$, we finally get the values of x and y in known quantities, and then the ellipse may easily be described by known methods.

XV. QUESTION 195, *by Mr.* I. R.

A parabola, and a point within it, being given in poſition, it is required to deſcribe another parabola, that ſhall paſs through the given point, touch the given parabola in a given point, and have its axis parallel to the axis of the given parabola.

Solution, *by Mr.* Lowry.

Let ABC, (fig. 108, pl. 5,) be the given parabola, P the given point within it, and PDQ the required parabola, passing through P, touching the given parabola in the given point Q, and having its axis parallel to the axis of the given parabola. Bisect PQ in I, and draw IE parallel to the axes of the parabolas meeting the curve in D, and the common tangent to the parabolas at Q in E; then, since I is a given point, and the tangent QE is given by position, the line IE is given in magnitude; and, therefore, ID (= DE, Em. Par. 20,) is also given in magnitude: Hence the ordinate PQ, and the diameter ID, are given to describe the parabola.

Or we may determine the focus and directrix of the parabola, by the following method: let *f* be the focus, join P*f*, Q*f*, and PE, and draw the diameters PB, QF; then EP is a tangent to the parabola at P, (Em. Par. 20, Cor. 2,) and bisects the angle BP*f* (36); also the tangent QE bisects the angle FQ*f*; therefore if P*f* and Q*f* be drawn to make the angles EP*f*, EQ*f* equal to the given angles BPE, EQF respectively, their intersection will determine the point *f*; and then, if PL be taken equal to P*f*, a line drawn through L perpendicular to PL, will be the directrix.

When P is the focus of the given parabola, (fig. 109, pl. 5,) and B the vertex, EP bisects the angle BPQ, and EQ bisects the angle PQF, (Em. Par. 8, Cor. 3;) therefore the point *f* is in the line PQ, and may be determined by drawing E*f* perpendicular to PQ.

Let the tangent at the vertex of the diameter ID, meet PE in H; draw HK parallel to PB, and join HB, BE; then PH = HE (20,) and because HK is parallel to PB, the angle PKH is = BPH = HPK; therefore the triangle PHK is isosceles, and PK = HK = ID. Again, because PE bisects the angle BPQ, BE is a tangent to the parabola at B, (36), and therefore perpendicular to PB; and since EP is bisected in H, the point H is the centre of a circle passing through the points P, B, E; therefore the angle BHE is = to twice the angle BPH or = to the angle BPQ. From hence we may infer, that if FQ be drawn from the focus, to any point Q, in the parabola ABC, and PE bisect the angle BPQ, and BH be drawn to make the angle BHE = BPQ, and HK be parallel to PB; then PQ is = to four times PK, or to four times ID; and the parabola described with the diameter ID, and ordinate PQ, will touch the parabola ABC at Q.

This conclusion immediately leads to the solution of the following problem: To determine the curve that touches all the parabolas described by a projectile from the same point, with the same initial velocity at different angles of elevation in the same vertical plane.

Let P be the initial position of the projectile, PE its direction, and PB the impetus, or the height to which the projectile would ascend in a vertical direction, with the given initial velocity.

Draw BH to make the angle BHE = BPQ, and draw HK parallel to PB; then it is shewn, by the writers on projectiles, (see Dalby's Course of Mathematics, Vol. 2, Art. 335,) that the parabola described by the projectile, will meet the line PQ, at a distance from P equal to four times PK: but it is proved above, that four times PK is equal to PQ, therefore the parabola described by the projectile, meets PQ in the point Q, where that line meets the parabola ABC. Also the greatest height of the projectile, or the diameter ID, is equal to HK, (ibid;) therefore the parabola described by the projectile, is the same as the parabola PDQ described above, and consequently the projectile will touch the parabola ABC at Q. And as a like point of contact may be found for any other angle of elevation, it is obvious that each parabola described by the projectile, will touch the parabola ABC; and, consequently, the curve required is the parabola ABC, whose *latus rectum* is = to four times the impetus.

COR. 1. The triangles BPE, PE*f*, are evidently equal in every respect; therefore, $Pf = PB$, and BE is the directrix of all the parabolas described by the projectile.

COR. 2. A circle described with the centre P, and radius PB, or (Pf,) will be the locus of the foci of all these parabolas.

COR. 3. If we conceive the parabola ABC, to revolve round the axis PB, so as to generate a parabolic superficies, it is obvious that this superficies will touch all the parabolas that can possibly be described by the projectile from P, with the given impetus, and in any direction whatever: and the locus of the foci will then be a spherical superficies.

Solutions were also received from Messrs. Cunliffe & O'Riordan.

XVI. QUESTION 196, *by Mr.* WILLIAM SLENDER.

Let there be a right line and a point given by poſition in the ſame vertical plane: and ſuppoſe a flexible ſtring of a given length, conſidered as without weight, to be faſtened at one end to the given point, and a ſmall heavy body or bead to ſlide freely thereon; required the locus of the bead, whilſt the other end of the ſtring is carried ſlowly along the right line given by poſition?

SOLUTION, *by Mr.* CUNLIFFE.

Let P, (fig. 110, pl. 5,) be the given point to which one end of the string PQR is fastened; SR the right line given by position, in the same vertical plane with P, and along which the other end R of the string, is slowly carried: also, let Q be the place of the bead corresponding with the point R.

Produce RS to meet PA, parallel to the horizon, in A, and draw PS perpendicular to PA, cutting RS in S. Produce PQ to r, so that $Qr = QR$, and draw the lines nr, NQ, and mR, parallel to PA, and terminating in PS in the points n, N, and m respectively: also, join Rr, and produce NQ to meet it in v.

Now the end R of the string is, by the question supposed to move very slowly along SR, and consequently, the bead Q will slide slowly along the string; therefore the bead Q may be imagined to be in equilibrio at every corresponding point R, upon the line SR. In the last mentioned circumstance it is well known, that $\angle PQN = \angle rQv = \angle RQv$, therefore Qv always bisects, and is perpendicular to Rr, and consequently Rr is parallel to PSN.

Put $PS = a$, $PA = b$, the length of the string $PQR = Pr = l$, $PQ = r$; tang. $\angle NPQ = t$, to radius 1, then its secant =

$\surd(1 + t^2)$, and by the principles of trigonometry, $\surd(1 + t^2) : t :: l : lt \div \surd(1 + t^2) = nr = m\mathrm{R}$, $\surd(1 + t^2) : 1 :: l : l \div \surd(1 + t^2) = \mathrm{P}n$; also, $\surd(1 + t^2) : 1 :: r : r \div \surd(1 + t^2) = \mathrm{PN}$. And because N$v$ bisects Rr, or Nm = Nn, Pm = 2PN — Pn, and therefore, Sm = 2PN — Pn — PS = $(2r - l) \div \surd(1 + t^2) - a = \left\{ 2r - l - a\surd(1 + t^2) \right\} \div \surd(1 + t^2)$. Again, by reason of similar triangles, APS, RmS, PA : PS :: mR : Sm, or

$b : a :: lt \div \surd(1 + t^2) : \left\{ 2r - l - a\surd(1 + t^2) \right\} \div \surd(1 + t^2)$; whence $r = \frac{1}{2}alt \div b + \frac{1}{2}a\surd(1 + t^2) + \frac{1}{2}l = \frac{1}{2}a(nt + \surd(1 + t^2)) + \frac{1}{2}l$, or $r^2 = \frac{1}{4}a^2(n^2t^2 + 2nt\surd(1 + t^2) + 1 + t^2) + \frac{1}{2}al(nt + \surd(1 + t^2)) + \frac{1}{4}l^2$, where $n = l \div b$.

When $t = 0$, $r = \frac{1}{2}(l + a)$, which determines the point where the curve cuts PS.

From the preceding equation of the curve, an equation expressed in terms of its own co-ordinates, with respect to the line PSN, as an axis may be easily derived. In order to this end, put PN = x, and NQ = y, then $r = \surd(x^2 + y^2)$, and we shall very evidently have

$x : y :: 1 : y \div x = t$, $x : \surd(x^2 + y^2) :: 1 : \surd(x^2 + y^2) \div x = \surd(1 + t^2)$; Now by means of these exterminating r and t out of the said equation, and it becomes $\surd(x^2 + y^2) = \frac{a}{2}\left(\frac{ny}{x} + \frac{\surd(x^2 + y^2)}{x}\right) + \frac{l}{2}$; whence $(2x - a)\surd(x^2 + y^2) = any + lx$, which is an equation for a line of the fourth order.

The area may be determined as follows:

Let z denote the circular arc, radius 1, and centre P, intercepted between PS and PQ; then we shall have $1 : r :: \dot{z} : r\dot{z}$ = the fluxion of the circular arc, radius r, and centre P, intercepted between PS and PQ, and hence $r \cdot \frac{r\dot{z}}{2} = \frac{r^2\dot{z}}{2} = \frac{a^2}{8} \times \frac{\dot{t}}{1 + t^2}(n^2t^2 + 2nt\surd(1 + t^2) + 1 + t^2) + \frac{al}{4} \cdot \frac{\dot{t}}{1 + t^2} \times (nt + \surd(1 + t^2)) + \frac{l^2}{8} \cdot \frac{\dot{t}}{1 + t^2}$ = the fluxion of the area of the space comprehended within the lines PS, PQ, and the curve; and the correct fluent of this expression is

$$\frac{l^2 - a^2n^2}{8} \cdot z + \frac{a^2t}{8}(n^2 + 1) + \frac{a^2n}{4}\surd(1 + t^2) + \frac{aln}{8} \times \text{h. l.}(1 + t^2) + \frac{al}{4} \times \text{h. l.}(t + \surd(1 + t^2)) - \frac{a^2n}{4}.$$

When $l = an = al \div b$, the quantity $\frac{l^2 - a^2n^2}{8} \times z$, will manifestly disappear out of the general expression for the area, which indicates, that when the line SR makes an angle of 45° with the horizon, the area is expressible independent of circular arcs; that is, the area in that case will be expressed by algebraic quantities and hyperbolic logarithms.

Solutions to this question were also received from Messrs. O'Riordan *and* Swale.

XVII. QUESTION 197, *by Mr.* IVORY.

If there be two concentric ellipses, similar to one another, and similarly posited, the proportion of the axes being that of $\sqrt{2}$ to 1; then, if from any point in the periphery of the inner ellipse, any right line be drawn to terminate in the periphery of the outer ellipse, the rectangle under the segments of that line, is equal to the square of that semi-diameter of the inner ellipse, which is parallel to the right line. Required the demonstration?

SOLUTION, *by Mr.* I. H. SWALE.

When the axes of the ellipses have the proportion in the question, the outer ellipse will be the locus of the angles of the parallelogram, formed by drawing tangents at the extremities of any two conjugate diameters of the inner ellipse, as is shewn at page 237, vol. 1. Hence, if any line CPD, (fig. 111, pl. 5,) be drawn to touch the inner ellipse at P, and meet the outer one in C and D, and the diameter AB, of the inner ellipse, be drawn parallel to CD; then AB will be = CD, and AQ = QB = CP = DP.

Now let any line be drawn through P, to meet the outer ellipse in K and L, and let QR be the semi-diameter, parallel to KL: Then, by a well known property of the sections, the rectangle CP.PD is to the rectangle KP.PL, as the square of the semi-diameter of the outer ellipse parallel to CD, to the square of the semi-diameter of the same ellipse parallel to KL; that is, because of the similarity of the ellipses, as AQ^2 to QR^2: but CP, PD is = to AQ^2, therefore KP.PL is = to QR^2.

This property is remarkable for its analogy to a property of the asymptotes of a hyperbola; for as the outer ellipse is the locus of the angles of all the parallelograms circumscribing the inner ellipse; so the asymptotes of a hyperbola are the *loci* of the angles of all the parallelograms formed by tangents drawn to the opposite and conjugate hyperbolas: And if any line be drawn through a point P in a hyperbola, to meet the asymptotes in K and L, the rectangle KP.PL is equal to the square of the semi-diameter parallel to CL, as in the ellipſe.

It was also answered by Dr. O'Riordan.

XVIII. QUESTION 198, *by* Mechanicus.

Required the nature of the curve along which a heavy body deſcending by the force of gravity, with a given initial celerity, shall preſs upon the curve at any point with a force reciprocally proportional to the radius of curvature at that point?

Solution, *by Mr.* Lowry.

Suppose that ABC, (fig. 112, pl. 5,) is the curve required, and that the body begins to descend from B, with the same velocity that it would have acquired by falling through the vertical height FB.

Let C be any point in the curve, and draw BD perpendicular to the vertical line CD. Put BD $= x$, DC $= y$, arch BC $= s$, FB $= a$, the radius of curvature at C $= r$, the velocity of the body at C $= v$, and the accelerative force of gravity $= g$.

Then the whole pressure on the curve at C, (the mass of the body being unit,) is $g \times \frac{\dot{x}}{\dot{s}} + \frac{v^2}{r}$, the part $g \times \frac{\dot{x}}{\dot{s}}$ being the normal pressure arising from gravity, and the part $\frac{v^2}{r}$, the pressure arising from the centrifugal force; but since the velocity of the body at C, is the same as would be acquired by descending through the height BF + CD or $a + y$, therefore $v^2 = 2g\,(a + y)$, and the whole pressure $= g\,(\frac{\dot{x}}{\dot{s}} + \frac{2(a + y)}{r})$.

But by the question, the pressure is always reciprocally proportional to the radius of curvature, or = to $\frac{n}{r}$, where n is a constant quantity; therefore $g\,(\frac{\dot{x}}{\dot{s}} + \frac{2(a + y)}{r}) = \frac{n}{r}$, or putting $2a - m = 2b$, $m = n \div g$; and multiplying by r, we have $r \times \frac{\dot{x}}{\dot{s}} + 2(b + y) = 0$.

Now when $\dot{s}$ is constant, the general expression for the radius of curvature, is $\frac{\dot{s}\dot{y}}{\ddot{x}}$, which being put in the last equation instead of r, we have $\frac{\dot{x}\dot{y}}{\ddot{x}} + 2(b + y) = 0$, or $\dot{x}\dot{y} + 2(b + y)\ddot{x} = 0$; and dividing by $2(b + y)^{\frac{1}{2}}$, we have

$$\frac{\dot{x}\dot{y}}{2(b + y)^{\frac{1}{2}}} + (b + y)^{\frac{1}{2}}\ddot{x} = 0.$$

Take the fluents, supposing $\dot{y}$ to be constant, and then $(b+y)^{\frac{1}{2}}\dot{x} = A^{\frac{1}{2}}\dot{y}$, where $A^{\frac{1}{2}}$ is a constant quantity necessary for completing the integral, and which must be determined from the particular data of the problem.

Let the last equation be divided by $(b + y)^{\frac{1}{2}}$, and the fluent taken, then $x = 2A^{\frac{1}{2}}(b + y) + C$, an equation to the common parabola.

In this equation, when $y = 0$, x is $= 2A^{\frac{1}{2}}b^{\frac{1}{2}} + C$, but x should then be $= b$, therefore C is $= -2A^{\frac{1}{2}}b^{\frac{1}{2}}$, and $x = 2A^{\frac{1}{2}}\left\{(b + y)^{\frac{1}{2}} - b^{\frac{1}{2}}\right\}$, or $(x + 2A^{\frac{1}{2}}b^{\frac{1}{2}})^2 = 4A(b + y)$.

Hence it is manifest, that if 4A be the parameter to the axis AE, and BH be taken = to b, the line HA drawn parallel to CE, will meet the parabola at A, the vertex of the axis.

To determine the value of the parameter, let AI be taken on the axis = to HF or $\frac{1}{2}m$; then it is obvious that the velocity, (and consequently the pressure) at any point of the curve, will be the same, whether the body begins to descend from A, with the velocity due to the height AI, or from B, with the velocity due to the height FB.

Now $\frac{\dot{x}}{\dot{s}}$ is = the sine of the angle which the tangent makes with the axis of the curve, and therefore at the vertex of the parabola $\frac{\dot{x}}{\dot{s}}$ is $= 1$, and the normal pressure $= g$. Also the radius of curvature at the vertex, is equal to half the parameter or $\frac{1}{2}p$, and the centrifugal force $= \frac{v^2}{r} = \frac{2g \times AI}{\frac{1}{2}p} = \frac{2gm}{p}$; but since the centre of curvature is on the contrary side of the curve with the body, it is evident that the centrifugal pressure is opposed to the normal pressure, and consequently the whole pressure at A, is $g - \frac{2gm}{p}$, which by the question is $=$ to $\frac{n}{r} = \frac{2n}{p}$; therefore p is $= 4m$.

The truth of this conclusion may easily be verified by a synthetical process; for let the abscissa AE, be put $= v$, and the ordinate CE $= u$; then u^2 is $= pv$, $\dot{u} = \frac{p^{\frac{1}{2}}\dot{v}}{2v^{\frac{1}{2}}}$, $\dot{s} = \frac{\dot{v}}{2}\sqrt{\frac{4v+p}{v}}$, and $\frac{\dot{u}}{\dot{s}} = \frac{p^{\frac{1}{2}}}{\sqrt{(4v+p)}} = \frac{p^{\frac{1}{2}}(4v+p)}{(4v+p)^{\frac{3}{2}}}$:

But (Simpson's Fluxions, pa. 76, vol. 1.) $\frac{(4v+p)^{\frac{3}{2}}}{2p^{\frac{1}{2}}} = r$; therefore, $\frac{\dot{u}}{\dot{s}} = \frac{4v+p}{2r}$, and the normal pressure is $= \frac{g(4v+p}{2r}$: Also the centrifugal force $\frac{v^2}{r} = \frac{2g(v+\frac{1}{2}m)}{r} = \frac{g(4v+2m)}{2r}$; therefore the whole pressure is $= \frac{g(4v+p)}{2r} - \frac{g(4v+2m)}{2r} = \frac{g(p-2m)}{2r} = \frac{gm}{r}$.

It is evident from hence, that if the body begins to descend from A, with any given velocity, the pressure at any point of the parabola, will be equal to a certain quantity divided by the radius of curvature at that point.

Messrs. O'Riordan *and* Swale *answered this question.*

XIX. QUESTION 199, *by Mr.* CUNLIFFE.

Find a ſeries of numbers, every one of which is diviſible into three ſuch parts, that if to the ſquare of each part the product of the other two be added, the three ſums thence ariſing ſhall all be rational ſquares?

SOLUTION, *by Mr.* CUNLIFFE, *the Proposer.*

Let n denote a number capable of being divided according to the conditions of the question; and let x, y, and z, denote the required parts thereof. Then by the question, $yz+x^2$, $xz+y^2$, and $xy+z^2$ are all to be rational squares: put $z=4(x+y)$, and then

$yz+x^2=x^2+4xy+4y^2=(x+2y)^2=$ a square,
$xz+y^2=4x^2+4xy+y^2=(2x+y)^2=$ a square,
$xy+z^2=16x^2+33xy+16y^2=$ a square.

Assume $4ry-4x$, for the root of the last square, that is, put $16x^2+33xy+16y^2=(4ry-4x)^2=16r^2y^2-32rxy+16x^2$, which gives $x=16y(r^2-1)\div(32r+33)$, and $z=4(x+y)=4y(16r^2+32r+17)\div(32r+33)$; also $x+y+z=5(x+y)=5y(16r^2+32r+17)\div(32r+33)$, which put $=n$: whence
$y=n(32r+33)\div5(16r^2+32r+17)$,
$x=16y(r^2-1)\div(32r+33)=16n(r^2-1)\div5(16r^2+32r+17)$,
and $z=4y(16r^2+32r+17)\div(32r+33)=\frac{4}{5}n$.

Hence it appears, that any number whatever, is capable of being divided according to the conditions of the question, and consequently, any series of numbers may be so divided, which shews the question to be very unlimited indeed. But if we confine our attention to such a series of numbers as are capable of being divided into three integral parts of the kind mentioned, we shall find the scope of the question more circumscribed. Numbers of the last mentioned kind are contained in the form $n=5(16r^2+32r+17)$, where r is any whole number at pleasure; and the three required parts of this number are $x=16(r^2-1)$, $y=32r+33$, and $z=4(16r^2+32r+17)$, the reason of which is obvious from barely inspecting the foregoing conclusions.

Dr. O'Riordan *also answered this question.*

XX. QUESTION 200, *by* HYPATIA.

Required the nature of the curve, that is, the locus of the point in which a perpendicular from the centre of an ellipſe meets a tangent to the ellipſe. Required alſo the rectification of the curve and its quadrature?

SOLUTION, *by Mr.* LOWRY.

Let AB, (fig. 113, pl. 5,) be the transverse, and DH the conjugate axis, C the centre, and F one of the foci of the ellipse. Also let P be the point where a perpendicular from the centre, meets a tangent drawn to the ellipse, at any point Q. Through the focus F, draw FE perpendicular to the tangent, meeting it in E; complete the rectangle CPEG, and draw CE; then CE is equal to the semi-transverse CB. (Emerson's Conic Sections, B. I. P. 20. Cor. 2.)

Put CB $= a$, CF $= e$, CP $= y$, the angle BCP, (or its measure, the arc ON, reckoned on the circle whose radius CN is 1,) $= x$, and the area of the space BCPB $= s$.

Then because CP is parallel to GE, the angle CFG is equal to the angle BCP $= x$; therefore CG $=$ CF $\sin x = e \sin x$, and EG (CP) $= \sqrt{(\mathrm{CE}^2 - \mathrm{CG}^2)} = \sqrt{(a^2 - e^2 \sin^2 x)}$,

$$\text{or } y = \sqrt{(a^2 - e^2 \sin^2 x)},$$

the equation of the curve expressed in terms of the variable line CP, and the variable angle BCP.

From this equation it appears, that when x is either $=$ to 0, or to two right angles, y is $=$ to a, and when x is either a right angle, or three right angles, $\sin^2 x$ is $= 1$, and $y = \sqrt{(a^2 - e^2)}$ $=$ CD or CH. Therefore it is evident that the curve will touch the ellipse in the points B, D, A, H, and that it will consist of four quadrantal spaces, equal and similar to the space CDPB.

Let p be a point in the curve, indefinitely near to P, and draw Cp, meeting the circle, whose radius is CN in n: also draw pI perpendicular to CP.

Then because On is the fluxion of the angle BCP, or $= \dot{x}$, pI is $=$ PC $\times \dot{x}$; therefore the triangle CpP ($= \frac{1}{2}$PC.pI) or the fluxion of the area BCPB is $= \frac{1}{2}$PC $\times \dot{x} = \frac{1}{2} y^2 \dot{x} = \frac{1}{2}(a^2 - e^2 \times \sin^2 x)\,\dot{x}$; that is $\dot{s} = \frac{1}{2} a^2 \dot{x} - \frac{1}{2} e^2 \sin^2 x \dot{x}$, and taking the fluents, we have

$$s = \tfrac{1}{2} a^2 x - \tfrac{1}{4} e^2 x - \tfrac{1}{8} e^2 \sin 2x,$$

which needs no correction.

And when x is a right angle, s is $= \frac{1}{2} p(a^2 - \frac{1}{2} e^2)$, p being the fourth part of the periphery of a circle whose radius is 1, therefore the whole area, or four times the space CDPB is

$$= 2p(a^2 - 2e^2).$$

Again, let the arc PB be denoted by z, and then $\dot{z}$ is $= \sqrt{(p\mathrm{I}^2 + \mathrm{IP}^2)}$: But pI is $= y\dot{x} = \dot{x}\sqrt{(a^2 - e^2 \sin^2 x)} = \dot{x}\,(a^2 - e^2 \sin^2 x) \div \sqrt{(a^2 - e^2 \sin^2 x)}$, and IP $= \dot{y} = \dot{x}\,(e^2 \sin x \cos x) \div \sqrt{(a^2 - e^2 \sin^2 x)}$; and substituting these values of pI and and IP in the expression for $\dot{z}$, we have, after proper reduction,

$$\dot{z} = \frac{\dot{x}\sqrt{\left\{a^4 - (2a^2 - e^2)\,e^2 \sin^2 x\right\}}}{\sqrt{(a^2 - e^2 \sin^2 x)}};$$

or putting $q = a^4 \div (2a^2 - e^2)\,e^2$, $p = a^2 \div e^2$, and $r^2 = 2a^2 - e^2$

$$\dot{z} = \frac{r\dot{x}\sqrt{(q - \sin^2 x)}}{\sqrt{(p - \sin^2 x)}}.$$

The fluent of this expression, may, in some particular cases, be expressed by elliptic arcs, but in general it is included in the formula H, or the third and most complex species of elliptic transcendentals, as will easily appear by making the proper transformation according to the method of Le Gendre.

If w be put $= q - \sin^2 x$, the expression for $\dot{z}$ becomes $\frac{rw^{\frac{1}{2}}\dot{x}}{\sqrt{(p-q+w)}}$. Or since $\sin x = \sqrt{(q-w)}$, $\cos x = \sqrt{(1-q+w)}$, and $\dot{x} = \frac{\text{flux.}\sin x}{\cos x} = \frac{-\dot{w}}{2\sqrt{\{(q-w)(1-q+w)\}}}$, we have

$$\dot{z} = \frac{-\frac{1}{2}rw^{\frac{1}{2}}\dot{w}}{\sqrt{(p-q+w)}\sqrt{\{(q-w)(1-q+w)\}}}.$$

This last expression agrees with formula 55, table 12, of Landen's Appendix, where the ingenious author has shewn how it may be resolved into other expressions that are more simple; but the integration by this method is exceedingly troublesome. Perhaps the easiest way of computing the fluent is at once to resolve the expression into a series without having recourse to any particular mode of transformation.

Messrs. Cunliffe, O'Riordan, *and* Swale, *also answered it.*

XXI. QUESTION 201, *by Mr.* IVORY.

If straight lines be inflected from the extremities of the transverse axis of an ellipse, or hyperbola, to any point in the curve, and perpendiculars, drawn to the inflected lines from the same point, be produced to cut the same axis, the part of the axis intercepted by the perpendiculars, will be constantly of the same magnitude. Required the demonstration?

SOLUTION, *by Mr.* I. H. SWALE.

Fig. 114, 115, pl. 5. Let AB be the transverse axis of the ellipse or hyperbola, and let the lines AP, BP be inflected to any point P in the curve. Draw PE perpendicular to AP, and PD perpendicular to BP, then by the question, ED is a constant magnitude.

Draw PC perpendicular to AB, then (Euc. VI. 8.) $PC^2 = AC.CE = BC.CD$, therefore $AC : BC :: CD : CE$, and by composition or division, according as the curve is an ellipse or hyperbola, $AB : BC :: ED : CE$, or $AB : ED :: BC : CE :: AC.BC : PC^2$ $(AC.CE)$.

But by a well known property of the sections,

AB : latus rectum $:: AC.BC : PC^2$;

Therefore DE is equal to the latus rectum.

A similar property belongs to the parabola; for if a line be drawn from the vertex A, to any point P in the curve, and PE be drawn perpendicular to AP, and PD perpendicular to the diameter at P; then the distance ED, on the axis, is equal to the latus rectum of the parabola.

If the lines are drawn from the extremities of the conjugate axis, to any point of the ellipse, or conjugate hyperbolas; then DE will still be a constant magnitude, being equal to a third proportional to the conjugate and transverse axes; and the demonstration of this case will be precisely the same as the preceding.

This question was also answered by Dr. O'Riordan.

XXII. QUESTION 202, *by Mr.* WALLACE, *R. M. College.*

Find the length of a curve, the nature of which is expreſsed by the equation

$$e^y = \frac{e^x + 1}{e^x - 1},$$

where x and y denote the co-ordinates, and e the number of which the hyperbolic logarithm is unity?

SOLUTION, *by Mr.* IVORY.

Put $z = e^x$, then $e^y = \frac{e^x + 1}{e^x - 1} = \frac{z + 1}{z - 1}$: therefore $x = \log. z$, and $y = \log. \frac{z + 1}{z - 1}$; or, if we put $u = \frac{z + 1}{z - 1}$, then $(z - 1)(u - 1) = 2$, and $x = \log. z$, and $y = \log. u$. When z increases from 1 to ∞, it is obvious that u decreases from ∞ to 1: therefore when x, or $\log. z$, increases from 0 to ∞, it follows that y, or $\log. u$, decreases from ∞ to 0. Hence it is manifest, that the curve has two asymptotes, AB and AC, (fig. 116, pl. 5,) passing through the origin of the coordinates, and parallel to them. In the case when $z = u$, we have $z = u = \sqrt{2} + 1$, and $x = y = \log. (\sqrt{2} + 1)$; which determines the point D where the curve cuts the line AD that bisects the angle of the asymptotes. And because z and u are interchangeable in the equation $(z - 1)(u - 1) = 2$, therefore the two branches of the curve, on opposite sides of this line, are similar and equal.

If, in the equation $(z - 1)(u - 1) = 2$, z be taken less than 1, (in which case x, or $\log. z$, would be negative), then u would be negative, and so y, or $\log. u$ would be the logarithm of a negative quantity, which is an absurd expression: therefore, although four curves, all similar and equal, may be described in the four angles of the asymptotes, yet these four curves are not connected by the law of continuity as two opposite hyperbolas are. For the equation that applies to one of the four curves, does not apply to any of the rest, by giving to the variable quantities of the equation all possible values from 0 to $+\infty$ on one hand, and from 0 to $-\infty$ on the other hand, while the origin of these variable quantities remains fixt.

The area ABMN, or curvilineal space between the asymptote and the ordinate MN, or y, $= \int y\,dx = \int \frac{dz}{z} \log. \frac{z+1}{z-1}$ $= \int \frac{dz}{z} \log. \frac{1+\frac{1}{z}}{1-\frac{1}{z}} = C - 2(\frac{1}{z} + \frac{1}{3^2 . z^3} + \frac{1}{5^2 . z^5}$ + &c.): where $C = 2(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2}$ + &c.). The whole curvilineal space between the curve and the two asymptotes, is $= C = 2(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2}$ + &c.): therefore half this space, or the space ABND, bounded by the line that bisects the angle of the asymptotes, either asymptote and the corresponding branch of the curve, is $= \frac{1}{2}C = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2}$ + &c. to this let the area of the right-angled triangle ADE, comprehended by the co-ordinates that correspond to the middle point of the curve, or $\frac{1}{2}\log.^2 (\sqrt{2} + 1)$, be added, and the whole area, between the asymptote and the ordinate DE, drawn parallel to the asymptote from the middle point of the curve, is $= \frac{1}{2}C + \frac{1}{2}\log.^2 (\sqrt{2} + 1)$: but when $z = \sqrt{2} + 1$, then $\frac{1}{z} = \sqrt{2} - 1 = a$, and the area ABDE $= C - 2(a + \frac{a^3}{3^2} + \frac{a^5}{5^2}$ + &c.): therefore, by equating the two values of the area ABDE, we get, $1 + \frac{1}{3^2} + \frac{1}{5^2}$ + &c. $= 2(a + \frac{a^3}{3^2} + \frac{a^5}{5^2}$ + &c.) $+ \frac{1}{2}\log.^2 (\sqrt{2} + 1)$.

(Vide Landen's Memoirs, pages 116, 117).

The length of an arch of the curve, as ND, $= \int \surd(dx^2 + dy^2)$ $= \int \surd(\frac{dz^2}{z^2} + \frac{4dz^2}{(z^2 - 1)^2}) = \int - dz \,.\, \frac{z^2 + 1}{z^2 - 1}$, (because the arch DN increases when z decreases) $= -\int dz + \int \frac{dz}{z+1}$ $- \int \frac{dz}{z - 1} = C - z + \log. \frac{z + 1}{z - 1} = \surd 2 + 1 - \log.$ $(\surd 2 + 1) - e^x + y$. Hence the difference of the ordinate MN, (or y,) and the arch ND $= \surd 2 + 1 - \log. (\surd 2 + 1)$ $- z = \surd 2 + 1 - \log. (\surd 2 + 1) - e^x$: and the excess of the whole curve Dn above the asymptote AB, both continued indefinitely, $= \surd 2 + 1 - \log. (\surd 2 + 1) - 1 = \surd 2 - \log.$ $(\surd 2 + 1)$.

If we take $z = \surd 2 + 1 - \log. (\surd 2 + 1)$, or $x = \log.$ $\{\surd 2 + 1 - \log. (\surd 2 + 1)\}$, a point will be determined in the curve such, that the arch DN = ordinate NM.

Messrs. O'Riordan *and* Toplis *sent solutions to this question.*

XXIII. QUESTION 203, *by Mr.* CUNLIFFE.

To find three numbers ſuch, that the ſum of every two ſhall be a biquadratic, and the ſum of all the three a ſquare number?

SOLUTION, *by Mr.* CUNLIFFE, *the Proposer.*

Let x, y, and z, denote the three numbers, and by the question, put $x + y = a^4$, $x + z = b^4$, and $y + z = c^4$; half the sum of these equations is $x + y + z = \frac{1}{2}(a^4 + b^4 + c^4)$ which is to be a square by the question. Let each of the assumed equations be subtracted from the preceding, and there will be had

$x = \frac{1}{2}(a^4 + b^4 - c^4)$, $y = \frac{1}{2}(a^4 + c^4 - b^4)$, and $z = \frac{1}{2}(c^4 + b^4 - a^4)$; whence it appears, that the question is reduced to finding three biquadratic numbers, half the sum of which shall be a square. And that the required numbers may be all positive, it is necessary that the sum of any two of the three biquadratic numbers must be greater than the remaining one. Now $m + n$, m and n, are the roots of three biquadratic numbers, half the sum of which will be a square, viz. $(m^2 + mn + n^2)^2$. But the sum of the two last, viz. $m^4 + n^4$, is manifestly less than the remaining one

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one, viz. $(m+n)^4 = m^4 + 4m^3n + 6m^2n^2 + 4mn^3 + n^4$, and therefore positive numbers to answer the conditions of the question cannot from thence be had.

In order to obtain positive numbers, put $m + n + sv = a$, $m + rv = b$, and $n + v\,(r+s) = c$: then

$$x + y + z = \tfrac{1}{2}(a^4+b^4+c^4) = \tfrac{1}{2}\left\{(m+n+sv)^4+(m+rv)^4+(n+v\,(r+s))^4\right\}$$

$$= (m^2+mn+n^2)^2 + 2v \times \left\{(m+n)^3 \times s + m^3r + n^3 \times (r+s)\right\}$$

$$+ 3v^2 \times \left\{(m+n)^2 \times s^2 + m^2r^2 + n^2 \times (r+s)^2\right\}$$

$$+ 2v^3 \times \left\{(m+n) \times s^3 + mr^3 + n \times (r+s)^3\right\} + v^4 \times (r^2+rs+s^2)^2,$$

which is to be a square by the question. In order to abridge the calculation, put $m^2 + mn + n^2 = \mathrm{A}$; $(m+n)^3 \times s + m^3r + n^3 \times (r+s) = \mathrm{B}$; $(m+n)^2 \times s^2 + m^2r^2 + n^2 \times (r+s) = \mathrm{C}$; $(m+n) \times s^3 + mr^3 + n \times (r+s)^3 = \mathrm{D}$; and $r^2 + rs + s^2 = \mathrm{A}'$; by means whereof, the foregoing expression to be made a square becomes

$$\mathrm{A}^2 + 2\mathrm{B}v + 3\mathrm{C}v^2 + 2\mathrm{D}v^3 + \mathrm{A}'^2v^4.$$

Assume $\mathrm{A} + (\mathrm{B} \div \mathrm{A})\,v + \mathrm{A}'v^2$ for its root, that is, put $\mathrm{A}^2 + 2\mathrm{B}v + 3\mathrm{C}v^2 + 2\mathrm{D}v^3 + \mathrm{A}'^2v^4 = (\mathrm{A} + (\mathrm{B} \div \mathrm{A})v + \mathrm{A}'v^2)^2 = \mathrm{A}^2 + 2\mathrm{B}v + (\mathrm{B}^2 \div \mathrm{A}^2)v^2 + \mathrm{AA}'v^2 + (2\mathrm{A}'\mathrm{B} \div \mathrm{A})\,v^3 + \mathrm{A}'^2v^4$,

which gives $v = \dfrac{\mathrm{B}^2 + \mathrm{A}^3\mathrm{A}' - 3\mathrm{A}^2\mathrm{C}}{2 \times (\mathrm{A}^2\mathrm{D} - \mathrm{AA}'\mathrm{B})}$.

And by writing $-\mathrm{A}'$ for A', we shall have

$$v = \frac{\mathrm{B}^2 - \mathrm{A}^3\mathrm{A}' - 3\mathrm{A}^2\mathrm{C}}{2 \times (\mathrm{A}^2\mathrm{D} + \mathrm{AA}'\mathrm{B})},$$

which is another value for v.

Two other values for v might be found by assuming $\mathrm{A}'v^2 + (\mathrm{D} \div \mathrm{A}')v \pm \mathrm{A}$ for the root of the preceding general expression to be made a square.

A solution was also received from Dr. O'Riordan.

XXIV. QUESTION 204, *by Mr.* LOWRY.

Of all ſemi-parabolas having the ſame area, required that along which a heavy body will deſcend, by the force of gravity, in the leaſt time poſſible?

SOLUTION, *by Mr.* LOWRY, *the Proposer.*

1. To find a general expression for the time of descent through an arc AB, (fig. 117, pl. 6,) of a parabola having its axis in the vertical line CB. Put BC $= a$, the height CD fallen through in

the time $t = x$, DE $= y$, the parameter $= 4p$, the arch AE $= s$, the velocity acquired in descending through the arch AE, or through the height CD, $= v$, and $g = 16\frac{1}{12}$ feet the space fallen through from rest in one second.

Then $\dot{s} \div \dot{t} = v = 2\sqrt{gx}$, or $\dot{t} = \dot{s} \div 2\sqrt{gx}$. But by the nature of the curve, $y^2 = 4p\,(a - x)$, or $y = 2\sqrt{p(a - x)}$, and $\dot{y} = -\dot{x}\sqrt{p} \div \sqrt{(a - x)}$; therefore $\dot{s} = \sqrt{(\dot{x}^2 + \dot{y}^2)} = \dot{x}\sqrt{(1 + p \div (a - x))} = \dot{x}\sqrt{((p + a - x) \div (a - x))}$, and

$$\dot{t} = \frac{\dot{x}}{2\sqrt{gx}} \times \sqrt{\frac{(p + a - x)}{a - x}}.$$

Put $x = az^2$, then $\dot{x} = 2az\dot{z}$, and $\dot{x} \div 2\sqrt{x} = a^{\frac{1}{2}}\dot{z}$, therefore, by substitution, $\dot{t} = \frac{a^{\frac{1}{2}}\dot{z}}{g^{\frac{1}{2}}}\sqrt{\frac{p + a - az^2}{a - az^2}} = \frac{\dot{z}}{g^{\frac{1}{2}}} \times$

$$\sqrt{\frac{p + a - az^2}{1 - z^2}} = \sqrt{\frac{p + a}{g}} \times \dot{z}\sqrt{\frac{1 - \frac{az^2}{p + a}}{1 - z^2}},$$

the fluent to be taken between the limits $z = 0$, and $z = 1$.

Let $e^2 = a \div (p + a)$, and E $=$ the quadrant of an ellipse, of which e is the excentricity, and 1 the semi-transverse axis;

then E is $= \int \dot{z}\sqrt{\frac{1 - \frac{az^2}{p + a}}{1 - z^2}}$, taken from $z = 0$, to $z = 1$,

and $t = \sqrt{((p + a) \div a)} \times$ E, the time of descent through the arc AB.

2. We have now to determine the particular values of p and a, so that the area ACB may be a constant quantity, and the time of descent through the arch AB a minimum, or, which is the same thing, that the time may be a constant quantity and the area ACB a minimum.

$$\text{Now } t \text{ is} = \sqrt{\frac{p + a}{a}} \times \mathrm{E} = \sqrt{\frac{p + a}{a}} \times \int \dot{z}\sqrt{\frac{1 - e^2z^2}{1 - z^2}}$$

$$\text{and the area ACB} = \tfrac{4}{3}p^{\frac{1}{2}}a^{\frac{3}{2}};$$

and taking the fluxion of the first equation, we have $(\dot{p} + \dot{a})\,\mathrm{E} \div 2\sqrt{(p + a)} + \dot{\mathrm{E}} \times \sqrt{(p + a)} = 0$. But to find $\dot{\mathrm{E}}$, or the fluxion of the integral $\int \dot{z}\sqrt{((1 - e^2z^2) \div (1 - z^2))}$, where e and z are variable, we must have recourse to the method of differencing *de curva in curvam* invented by Leibnitz. This method consists in finding the fluxion of such expressions as $\int \mathrm{P}\dot{z}$, where P is a function of e and z, the quantity e denoting the parameter, diameter, excentricity, &c. of a curve; and which is supposed to be constant in the course of the same curve, but to vary in passing from that curve to another which is immediately contiguous. The general expression for the fluxion of $\int P\dot{z}$, when found by this method, is $P\dot{z} + \dot{e}\int \dot{z}(\frac{\dot{P}}{\dot{e}})$, where $\dot{P}$ is taken on the supposition that e only is variable.

In the above expression, P is $= \sqrt{\frac{1 - e^2z^2}{1 - z^2}}$, $P\dot{z} = \dot{z} \times \sqrt{\frac{1 - e^2z^2}{1 - z^2}}$, and $\dot{P} = - \frac{e\dot{e}z^2}{\sqrt{(1 - z^2)(1 - e^2z^2)}}$; therefore $\dot{e}\int \dot{z}(\frac{\dot{P}}{\dot{e}}) = - \dot{e}\int \frac{\dot{z}ez^2}{\sqrt{(1 - z^2)(1 - e^2z^2)}} = - e\dot{e} \times \int \frac{z^2\dot{z}}{\sqrt{(1 - z^2)(1 - e^2z^2)}}$, and $\dot{E} = \dot{z}\sqrt{\frac{1 - e^2z^2}{1 - z^2}} - \dot{e}e \times \int \frac{z^2\dot{z}}{\sqrt{(1 - z^2)(1 - e^2z^2)}}$: but since E is the quadrant of the ellipse, or the whole fluent of $\dot{z}\sqrt{\frac{1 - e^2z^2}{1 - z^2}}$ from $z = 0$, to $z = 1$, it is evident that the first part of the expression for $\dot{E}$ is $=$ to 0, and therefore $\dot{E}$ is $= -\dot{e}e\int \frac{z^2\dot{z}}{\sqrt{(1 - z^2)(1 - e^2z^2)}}$, (taken from $z = 0$, to $z = 1$).

Therefore $\frac{\dot{p} + \dot{a}}{2\sqrt{(p + a)}} \times E - \sqrt{(p + a)}\dot{e}e\int \frac{z^2\dot{z}}{\sqrt{(1 - z^2)(1 - e^2z^2)}} = 0$, or $(\dot{p} + \dot{a}) \times E - 2(p + a)\dot{e}e\int \frac{z^2\dot{z}}{\sqrt{(1 - z^2)(1 - e^2z^2)}} = 0$.

And taking the fluxion of the second equation, we have $(\dot{p} \div 2p^{\frac{1}{2}}).a^{\frac{2}{3}} + \frac{3}{2}a^{\frac{1}{2}}p^{\frac{1}{2}}\dot{a} = 0$, or $a\dot{p} = -3p\dot{a}$, and $\dot{p} = -(3p \div a)\dot{a}$; therefore $\dot{p} + \dot{a} = (1 - 3p \div a)\dot{a}$ But since e^2 is $= a \div (a + p)$, $1 \div e^2$ is $= 1 + p \div a$, or $4 - 3 \div e^2 = 1 - 3p \div a$, and $\dot{p} + \dot{a} = (4 - 3 \div e^2)\dot{a}$. Also $1 - e^2 = p \div (a + p)$, and $\dot{e}e = (\dot{a}p - a\dot{p}) \div 2(a + p)^2 = 2\dot{a}p \div (a + p)^2$, therefore $2(p + a)\dot{e}e = 4p\dot{a} \div (a + p) = 4(1 - e^2)\dot{a}$, and by substitution,

$$\left(4-\frac{3}{e^2}\right)\dot{a}\times E-4(1-e^2)\,\dot{a}\times\int\frac{z^2\dot{z}}{\sqrt{(1-z^2)\,(1-e^2z^2)}}=0,$$

$$\text{or }(4e^2-3)\,E-4(1-e^2)\times\int\frac{e^2z^2\dot{z}}{\sqrt{(1-z^2)\,(1-e^2z^2)}}=0.$$

$$\text{But }\frac{e^2z^2\dot{z}}{\sqrt{(1-z^2)(1-e^2z^2)}}=\frac{\dot{z}}{\sqrt{(1-z^2)\,(1-e^2z^2)}}-\dot{z}\sqrt{\frac{1-e^2z^2}{1-z^2}},$$

$$\text{therefore }(4e^2-3)\,E-4(1-e^2)\times\int\frac{\dot{z}}{\sqrt{(1-z^2)\,(1-e^2z^2)}}+4(1-e^2)E=0,$$

$$\text{or }E=4(1-e^2)\times\int\frac{\dot{z}}{\sqrt{(1-z^2)\,(1-e^2z^2)}}.$$

That is, in series, $1-\frac{1}{2^2}e^2-\frac{1}{2^2}\cdot\frac{3}{4^2}e^4-\frac{1}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5}{6^2}e^6-\&c.$

$$=4(1-e^2)\left(1+\frac{1}{2^2}e^2+\frac{1}{2^2}\cdot\frac{3}{4^2}e^4+\frac{1}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5}{6^2}e^6+\&c.\right.$$

$$\text{or }3=\left(3-\frac{1}{2^2}\right)e^2+\frac{1}{2^2}\left(\frac{7}{2^2}-\frac{3}{4^2}\right)e^4+\frac{1}{2^2}\cdot\frac{3^2}{4^2}\left(\frac{11}{3^2}-\frac{5}{6^2}\right)e^6+\&c.$$

$$\text{or }3=\frac{11}{2^2}e^2+\frac{1}{2^2}\cdot\frac{25}{4^2}e^4+\frac{1}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{39}{6^2}e^6+\frac{1}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}\cdot\frac{58}{8^2}e^8+\&c.$$

And by reverting the series, we may determine the value of e, and then a and p become known, when either the time or area is given.

The series for e may be found rather differently as follows: Let n be $=3{\cdot}14159$, &c.

$$\text{Then }E\text{ is }=\frac{n}{2}\left(1-\frac{1}{2^2}e^2-\frac{1}{2^2}\cdot\frac{3}{4^2}e^4-\frac{1}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5}{6^2}e^6-\&c.\right.$$

$$=\frac{n}{2}\,(1-Ae^2-Be^4-Ce^6-\&c.)$$

and $t=\sqrt{((a+p)\div g)}\times\frac{1}{2}n\,(1-Ae^2-Be^4-Ce^6-\&c.)$

Taking the fluxion, we have

$$\begin{aligned}0=&\left((\dot{p}+\dot{a})\div 2\sqrt{(p+a)}\right)\times(1-Ae^2-Be^4-Ce^6-\&c.)\\&+\sqrt{(p+a)}\,e\dot{e}\times(-2A-4Be^2-6Ce^4-8De^6-\&c.)\end{aligned}$$

and substituting for $\dot{p}+\dot{a}$, and $e\dot{e}$, their values, as found above, we have

$$\begin{aligned}0=&\,(4e^2-3)\,(1-Ae^2-Be^4-Ce^6-\&c.)\\&+4(1-e^2)\,(-2Ae^2-4Be^4-6Ce^6-\&c.)\end{aligned}$$

$$\text{or }3=\left.\begin{matrix}4\\+3A\\-8A\end{matrix}\right\}e^2+\left.\begin{matrix}+8A\\-4A\\+3B\\-16B\end{matrix}\right\}e^4+\left.\begin{matrix}+16B\\-4B\\+3C\\-24C\end{matrix}\right\}e^6+\&c.$$

That is, $3=(4-5A)\,e^2+(4A-13B)e^4+(12B-21C)e^6$

+

$+$ &c. or $3 = \frac{11}{2^2}e^2 + \frac{1}{2^2} \cdot \frac{25}{4^2}e^4 + \frac{1}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{39}{6^2}e^6 +$ &c, the same as before.

The problem may be resolved in a similar way, when the arch AP, or any function of a and p is given; and the process will not be materially different whatever the nature of the curve may be. In a circle, for example, if the length of the arch be given, we may find the radius so that the time of descent through the given arc, shall be a minimum. Or, which is the same thing, if a pendulum vibrates through a circular arch of a given length, we may find the length of the pendulum, so that the vibrations may be made in the least time.

Put r for the radius CD, (fig. 118, pl. 6,) or the length of the pendulum, BD = half the given arch, a = DE, and x = DI; then the time of descent through BD, is equal to the $\int \frac{r\dot{x}}{2\sqrt{((2rx - x^2)(a - x))}}$, between the limits $x = a$, and $x = 0$; and the arch BD $= \int \frac{r\dot{x}}{\sqrt{(2rx - x^2)}}$ between the same limits.

By making the fluxions of these expressions = to 0, we may after proper reduction, obtain the value of r in a series; or; since $t = \frac{n}{2}\sqrt{\frac{r}{2g}} \times \left\{ 1 + (\frac{1}{2})^2 \cdot \frac{a}{2r} + (\frac{1}{2} \cdot \frac{3}{4})^2 \cdot \frac{a^2}{2^2 r^2} + (\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6})^2 \cdot \frac{a^3}{2^3 r^3} \&c \right\}$

and the arch DB $= \sqrt{(2ra)} \times (1 + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{a}{2r} + \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{5} \cdot \frac{a^2}{2^2 r^2} + \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{6} \cdot \frac{5}{7} \cdot \frac{a^3}{2^3 r^3}$ &c.)

we may find r by taking the fluxions of these series and exterminating a, as we have done in the example above, but the process will be a little tedious.

XXV. QUESTION 205, *by* URSA MINOR.

Suppoſe a right cone to be cut by a plane, and the curved ſurface of the part cut off to be extended into a plane ſurface. Requi- red the nature of the plane curve, formed by the common ſection of the cone and cutting plane?

SOLUTION, *by Mr.* WALLACE, *R. M. College.*

Let V, (fig. 119, pl. 6,) be the vertex of the cone, APB its section by the plane, VC its axis meeting the section in C, AVB a section of the cone through its axis, and perpendicular to the section APB, ACB the common section of the two planes AVB and APB. Draw VP in the surface of the cone to P any point in the curve APB, and join CP. Let EQF be a section of the cone, perpendicular to its axis, (which section will of course be a circle,) and let FV (the distance of its circumference from the vertex,) be unity. Let this circle meet VB, VP in F and Q, and draw FD, QD to its centre. Put VC $= d$, the angle CVB or CVP $= \alpha$, the angle BCV $= \beta$. Also put the indeterminate line VP $= r$, and the indeterminate arch FQ $= \varphi$. Then because FD $= \sin \alpha$, the angle FDQ, (that is the angle contained by the planes BCV, PCV) will be $\varphi \div \sin \alpha = n\varphi$, (putting $n = 1 \div \sin \varphi$).

Now, considering the three planes VCB, BCP, VCP as forming a solid angle at C, and that the plane VCB is perpendicular to BCP, we have by spherics, tan VCP $=$ tan VCB $\div$ cos FDQ $= \tan \beta \div \cos n\varphi$. Draw PG perpendicular to VC; then in the right-angled triangle VPG, we have PG $=$ VP $\times$ sin PVG $= r \sin \alpha$, and VG $=$ VP $\times$ cos PVG $= r \cos \alpha$. And in the right-angled triangle PGC, we have CG $=$ PG $\times$ (1 $\div$ tan VCP) $= r \sin \alpha \cos n\varphi \div \tan \beta$. But VG $+$ GC $=$ VC $= d$; therefore $r \cos \alpha + r \sin \alpha \cos n\varphi \div \tan \beta = d$, and hence

$$r = \frac{\tan \beta}{\cos \alpha \tan \beta + \sin \alpha \cos n\varphi} \times d,$$

or putting $\sin \beta \div \cos \beta$ instead of $\tan \beta$,

$$r = \frac{\sin \beta}{\cos \alpha \sin \beta + \sin \alpha \cos \beta \cos n\varphi} \times d.$$

Conceive now the surface of the cone to be cut along the line VA, and extended into a plane, as in fig. 120, pl. 6. Then the surface will form the figure AVAPB, and the circle EFQ in fig. 119, will become an arc EFQE, in fig. 120, having its radius $= 1$. Therefore in the plane figure AVA, fig. 120, the nature of the curve BPA, is defined by the equation

$$r = \frac{\sin \beta}{\cos \alpha \sin \beta + \sin \alpha \cos \beta \cos n\varphi} \times d,$$

where α and β denote given angles, d a given line, and where φ denotes the indeterminate angle BVP, (VB being supposed given by position), and r the indeterminate line VP.

Because $d = (\sin(\alpha + \beta) \div \sin \beta) \times$ VB, the equation may be otherwise expressed thus

$$r = \frac{\sin(\alpha + \beta)}{\cos \alpha \sin \beta + \sin \alpha \cos \beta \cos n\varphi} \times \text{VB},$$

which again may be transformed to

$$r = \frac{\tan \beta + \tan \alpha}{\tan \beta + \tan \alpha \ \cos \varphi} \times \text{VB}.$$

If the section be a parabola, then $\varphi\alpha = \beta$, and the equation becomes simply,

$$r = \frac{2}{1 + \cos n\varphi} \times \text{VB}.$$

XXVI.

XXVI. QUESTION 206, *by Mr.* P. R.

Two magnetical poles being given in poſition; the force of each of which is ſuppoſed to be as the *m*th. power of the diſtance from it reciprocally; it is required to find a curve, in any point of which a needle (indefinitely ſhort) being placed, its direction, when at reſt, may be a tangent to the curve?

The following SOLUTION to this question, is taken from the Supplement to the Encyclopedia Britannica, Art. Magnetism, written by Dr. Robison, who there informs us that he received it from Mr. Playfair, of the University of Edinburgh.

1. Let A and B, (fig. 121, pl. 6,) be the poles of a magnet, C any point in the curve required; then we may suppose the one of these poles to act on the needle only by repulsion, and the other only by attraction, and the direction of the needle, when at rest, will be the diagonal of a parallelogram, the sides of which represent these forces. Therefore, having joined AC and BC, let AD be drawn parallel to BC, and make $\frac{1}{AC^m} : \frac{1}{BC^m} :: AC : AD$; join CD, then CDF will touch the curve in C.

2. Hence an expression for AF may be obtained. For, by the construction, $AD = \frac{AC^{m+1}}{BC^m}$, and since $BC : AD :: BF : FA$, and $BC - AD : AD :: AB : AF$, we have $AF = \frac{AB \times AC^{m+1}}{BC^{m+1} - AC^{m+1}}$.

3. A fluxionary expression for AF may also be found in terms of the angles CAB, ABC. In CF take the indefinitely small part CH, draw AH, BH, and from C draw CL perpendicular to AH and CK to BH: Draw also BG and AM at right angles to FH. Let the angles $CAB = \varphi$, and $CBA = \psi$; then $CAH = \dot{\varphi}$, and $CBH = -\dot{\psi}$; also, $CL = AC \times \dot{\varphi}$, and $CK = -BC \times \dot{\psi}$. Now $HC : CL :: AC : AM = \frac{AC^2 \times \dot{\varphi}}{HC}$; and for the same reason, $BG = -\frac{BC^2 \times \dot{\psi}}{HC}$. Therefore,

since AF : FB :: AM : BG, AF : FB :: $\frac{AC^2 \times \dot{\varphi}}{HC} : -\frac{BC^2 \times \dot{\psi}}{HC}$, and AF : AB :: $\sin\psi^2\dot{\varphi} : -\sin\psi^2\dot{\varphi} - \sin\varphi^2\dot{\psi}$;

wherefore, if $AB = a$, $AF = \frac{-a\sin\psi^2\dot{\varphi}}{\dot{\psi}\sin\varphi^2 + \dot{\varphi}\sin\psi^2}$.

4. If this value of AF be put equal to that already found, a fluxionary equation will be obtained, by the integration of which the curve may be constructed. Because $AF = \frac{AB \times AC^{m+1}}{BC^{m+1} - AC^{m+1}}$

and since $AC = \frac{a\sin\psi}{\sin(\varphi + \psi)}$, and $BC = \frac{a\sin\varphi}{\sin(\varphi + \psi)}$,

we have, by substitution, $AF = \frac{a\sin\psi^{m+1}}{\sin\varphi^{m+1} - \sin\psi^{m+1}} = -\frac{a\dot{\varphi}\sin\psi^2}{\dot{\psi}\sin\varphi^2 + \dot{\varphi}\sin\psi^2}$. Hence, $\sin\varphi^2 \times \dot{\psi}\sin\psi^{m+1} + \dot{\varphi}\sin\psi^{m+3} = -\sin\psi^2 \times \dot{\varphi}\sin\varphi^{m+1} + \dot{\varphi}\sin\psi^{m+3}$, and therefore, $\dot{\psi}\sin\psi^{m-1} = -\dot{\varphi}\sin\varphi^{m-1}$; and also

$\int\dot{\psi}\sin\psi^{m-1} + \int\dot{\varphi}\sin\varphi^{m-1} = C$.

5. These fluents are easily found, when *m* is any whole positive number.

If $m = 1$, we have $\dot{\psi} + \dot{\varphi} = 0$; and $\varphi + \psi = C$,

$m = 2$, we have $\dot{\psi}\sin\psi + \dot{\varphi}\sin\varphi = 0$; and $\cos\varphi + \cos\psi = C$; &c.

The first of above equations belongs to a segment of a circle described upon AB, which therefore would be the curve required if the magnetical force were inversely as the distances.

If the magnetical force be inversely as the square of the distance, that is, if $m = 2$, $\cos\varphi + \cos\psi$ is equal to a constant quantity. Hence, if, beside the points A and B, any other point be given in the curve, the whole may be described. For instance, let the point E, (fig. 122, pl. 6,) be given in the curve, and in the line DE which bisects AB at right angles. Describe from the centre A, a circle through E, viz. QER; then AD being the cosine of DAE to the radius AE, the sum of the cosines of φ and ψ will be every where (to the same radius) $= 2AD = AB$. Therefore to find E′, the point in which any other line AN, making a given angle with AB, meets the curve, draw from

N, the point in which it meets the circumference of the circle QER, NO, perpendicular to AB, so that AO may be the cosine of NAO, and from O toward A take OP = AB, then AP will be the cosine of the angle ABE′; so to find BE′, draw PQ perpendicular to AP, meeting the circle in Q; join AQ, and draw BE′ parallel to AQ, meeting AE′ in E′, the point E′ is in the curve. In this way the other points of the curve may be found.

XXVII. QUESTION 207, *by Mr.* WALLACE, *R. M. College.*

Shew that the rectification of the curve called the *line of ſines* depends upon that of an ellipſe, without employing the fluxionary calculus, or in any other way comparing the indefinitely little increments of the curves?

SOLUTION, *by Mr.* WALLACE, *the Proposer.*

Let AEB, (fig. 123, pl. 6,) be half the base of a cylinder, and AFB a section of the cylinder through C the centre of its base, making with it an angle of 45°, and which will be a semi-ellipse: Let CEF be a plane perpendicular to AB. From P, any point in APEB, the semi-circumference of the base, draw PQ in the surface of the cylinder perpendicular to the base, and meeting the ellipse AFB in Q. Through PQ, draw a plane perpendicular to AB, meeting the base in PD, and the plane AFB in QD. Then it is easy to see that DP = PQ, and consequently, that PQ is the sine of the arch AP. Hence it follows, that the portion of the cylindric surface contained between the semi-circle AEB, and semi-ellipse AFB would, if expanded into a plane, form a figure of sines, having for its base AEB, the semi-circumference of the of the cylinder. Now if it be remarked that CF : CA :: $\surd 2$: 1; it will be evident from what has been said, that the length of the line of sines is equal to half the perimeter of an ellipse whose axes are to one another as $\surd 2$ to 1, and whose conjugate axis is equal to the diameter of the circle from which the line of sines is conceived to be generated.

From what has been shewn, it is also easy to see that we can always express any arch of the line of sines by a corresponding arch of the same ellipse.

XXVIII. QUESTION 208, *by Mr.* IVORY.

Two points being given in a vertical plane, but not in the same horizontal line; it is required to determine the position of three inclined planes, such, that (supposing the body to pass from one plane to another, without any loss of velocity,) the time of descent may be less than the time on any other three planes, when the line of descent is of a given length ?

SOLUTION, *by Mr.* IVORY, *the Proposer.*

As the solution of this question seems to require complicated calculations, I shall be content with pointing out the method to be pursued. For the greater simplicity, the number of planes may be supposed to be two in place of three: then, preserving the notation in question 188, the time of the fall is $\frac{\varrho'}{a - x} \times (\sqrt{a} - \sqrt{x}) + \frac{\varrho}{x}\sqrt{x}$, which is to be a minimum; and the line of descent, is $\varrho + \varrho'$, which is to be of a given length, or equal to a given quantity, A. Let m be an indeterminate quantity, and having multiplied $\varrho + \varrho'$ by m, let the product be added to the function which is to be a minimum, and denote the sum by $f(x, y)$; that is, let $f(x, y) = \frac{\varrho'}{a - x} \times (\sqrt{a} - \sqrt{x}) + m\varrho' + \frac{\varrho}{x}\sqrt{x} + m\varrho$. The co-ordinates x and y, are next to be determined so as to make the function $f(x, y)$ a minimum: and, for this end, the equations $\left(\frac{df(x, y)}{dx}\right) = 0$, and $\left(\frac{df(x, y)}{dy}\right) = 0$, must be resolved, (see question 188). The values of x and y, thus found, besides involving the quantities given in the problem, will likewise contain the indeterminate quantity m: and being substituted in the equation $\varrho + \varrho' = A$, or $\sqrt{(x^2 + y^2)} + \sqrt{((a - x)^2 + (b - y)^2)} = A$, there will be obtained an equation which must be satisfied by taking m of a proper value.

If there be three inclined planes, the function which is to be made a minimum, will contain four variable co-ordinates; and a like number of equations for determining these will be obtained by taking the partial fluxions in that function; in other respects the same method of solution is to be pursued as shewn in the case of two inclined planes.

XXIX. QUESTION 209, *by* CURIOSUS.

Find the shortest distance between two given points on the surface of a spheriod ?

To this question we have received no answer.

XXX. PRIZE QUESTION 210, *by Mr.* LOWRY.

In a given ellipſe to inſcribe a polygon of a given number of ſides, ſo that each ſide may paſs through a given point?

SOLUTION, *by Mr.* LOWRY, *the Proposer.*

LEMMA. Let a, b, (fig. 124, pl. 6.) be two points in an ellipse, and let a circle be described on the conjugate axis XY; draw aI, bK parallel to the transverse axis meeting the circle in A and B, and XY in I and K. Join ab, AB, and from any point p, in ab, draw pm parallel to aI or bK, meeting AB in P, and XY in m; then pm, is to Pm, as the transverse axis to the conjugate axis.

For $a\text{I}^2 : b\text{K}^2 :: \text{XI.IY} = \text{IA}^2 : \text{XK.KY} = \text{KB}^2$;
therefore $a\text{I} : b\text{K} :: \text{IA} : \text{KB}$,
or $a\text{I} : \text{IA} :: b\text{K} : \text{KB} :: pm : \text{P}m$;
therefore $pm^2 : \text{P}m^2 :: a\text{I}^2 : \text{IA}^2 = \text{XI.IY}$: : square of the transverse : the square of the conjugate, or pm is to Pm, as the transverse axis to the coujugate axis.

Now let $abcde$, &c. (fig. 125, pl. 6,) be the polygon inscribed in the given ellipse, so that the sides ab, bc, cd, de, &c. may pass through the given points p, q, r, s, t, &c. respectively; and let a circle be described on the conjugate axis XY. Draw aA, bB, cC, dD, eE, &c. parallel to the transverse axis of the ellipse meeting the circle in the points A, B, C, D, E, &c. and join AB, BC, CD, DE, EF, &c. Draw pP, qQ, rR, sS, tT, &c. from the given points p, q, r, s, t, &c. parallel to the transverse axis, meeting the lines AB, BC, CD, DE, EF, &c. in P, Q, R, S, T, &c. and produce pP to meet XY in m. Then by the *lemma*, pm is to Pm, as the transverse axis to the conjugate axis; and since p is a given point, the distance pm is given; therefore Pm is given, and P is a given point. And in a similar way it is shewn that Q, R, S, T, &c. are given points. Wherefore it is evident that if a polygon ABCDEF, &c. be inscribed in the circle, so that its sides AB, BC, CD, DE, EF, &c. may pass through the given points P, Q, R, S, T, &c. the points a, b, c, d, e, &c. in the ellipse, may be found by drawing lines from the angular points of the polygon inscribed in the circle, parallel to the transverse axis of the ellipse.

To inscribe a polygon in a circle so that its sides shall pass through given points, is a problem that has been often resolved; and the solution that I shall give here is not materially different from others that have appeared before, but I shall endeavour to put it in rather a simpler point of view.

And first to inscribe a triangle ABC (fig. 126, 127, pl. 6,) in a given circle, so that its sides AB, BC, AC shall pass through the given points P, Q, R.

Join PQ, and make the rectangle HQP equal to the given rectangle CQB, and draw HC to meet the circle in K and join AK; then H is a given point, and the points B, C, H, P are in the circumference of the same circle; therefore the angle QHC is = CBP = AKH in fig. 126, or its supplement in fig. 127 (22.3), and AK is parallel to PQ. To the centre O, draw HO meeting the circle in N, and AK in I, and join CN. Make the rectangle HO.OP′ = to the square of the semi-diameter of the circle, and draw CP′ to meet the circle in G, and join KG; then P′ is a given point, and (Playfair's Euclid VI. P. F. Cor.) CN bisects the angle GCK; therefore the arcs KN, NG are equal, and consequently KG is perpendicular to HO: therefore the angle ACP′, or IKG, is the complement of the angle KIH, or equal to the difference between the given angle PHO and a right angle. Hence we have this

Construction: Find the points H and P′ as in the analysis, and on P′R describe a segment of a circle to contain an angle equal to the difference between the angle PHO and a right angle, and let it meet the given circle in C; draw QCB, CRA to meet the circle in B and A, then the line which joins the points B, A will pass through the given point P.

Demonstration. - Draw CH, CP′ to meet the circle in K and G, and join AK, KG and CN; then because HO.OP′ is = ON^2, KG is perpendicular to OH as is shewn in the analysis, and therefore the angle IKG or ACP′ is the complement of KIH. But by construction the angle ACP′ is the complement of PHO, therefore the angle HIK is equal to the angle PHO, and AK is parallel to PQ; therefore the angle QHC is = HKI = CBA. But because the rectangle HQP is = to the rectangle CQB, the points B, C, H, P are in the circumference of the same circle, therefore the angle QHC is = CBP, and consequently the angle CBA is = to the angle CBP; therefore the line BA passes through P.

The solution of this particular case, immediately leads to the solution of the general problem, when the polygon has any number of sides whatever.

Let the points HP′ (fig. 128, pl. 6,) be found as before for the given points P, Q; and join P′R, (the point R being that through which the side CD is to pass). On P′R, produced if necessary, find the point H′, so that the rectangle PRH′ may be equal to the given rectangle DRC, and produce OH′ to P″, so that the rectangle P″OH′ may be equal to the square of the semi-diameter, and join P″D meeting the circle in *m*; then H′ and P″ are given points, and because the construction for the points H′, P″ as related to the points P′ R; is the same as the construction for the points H, P′ as related to the points P, Q; it is evident from the preceding analysis that the angle *m*DG is the difference between the angle $P''H'P'$ and a right angle, and is therefore given; and it is shewn in the preceding case, that the angle ACG or ADG is given; therefore the angle ADm, or or ADP'' is given. Hence, when the number of sides of the polygon is four, or AD passes through a given point S', a circle described on $P''S'$, to contain an angle equal to the given one, will intersect the circle in D, and the construction will then be obvious.

If the polygon has more sides than four, join $P''S$ (S being the point through which the side DE is to pass), and find the point H'' in $P''S$, so that the rectangle $P''SH''$ may be equal to the given rectangle DSE, and in $H''O$ produced take the point P''', so that the rectangle $H''OP'''$ may be equal to the square of the semi-diameter, and join $P'''E$ meeting the circle in n: then H'' and P''' are given points, and by the foregoing analysis the angle $P'''Em$ is given, being equal to the difference between the angle $OH''P''$ and a right angle; therefore the angle mEn is given, and the angle AEm or ADP has been shewn to be given; therefore the angle AEn is given: and, if AE be the last side of the polygon, a circle described through the points $P'''T$ to contain the given angle will intersect the given circle in E. And in the same manner we may proceed for any polygon whatever, by joining the point last found, and the point through which the next side of the polygon is to pass, and making a similar construction to that which we have made above.

Mr. LOWRY *is requested to send to Mr.* GLENDINNING'S *for the Medal for solving the Prize Question.*

Since the preceding sheets were printed off, we have received the following solution to question XI. (191).

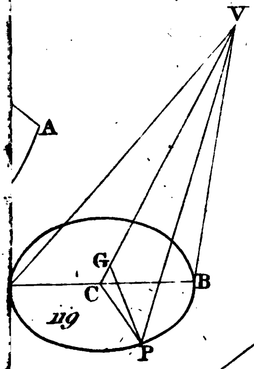
XI. QUESTION 191, *from the* CAMB. UN. CAL.

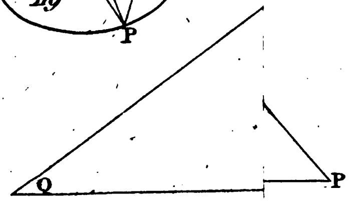
Bradley obſerved that every ſtar paſſed the meridian fartheſt ſouth when it came about ſix in the morning, whatever were its poſition with reſpect to the cardinal points of the ecliptic. Suppoſe then the place of a ſtar to be given; on what day will it paſs the meridian of London fartheſt to the ſouth; and at what hour will it paſs it on that day?

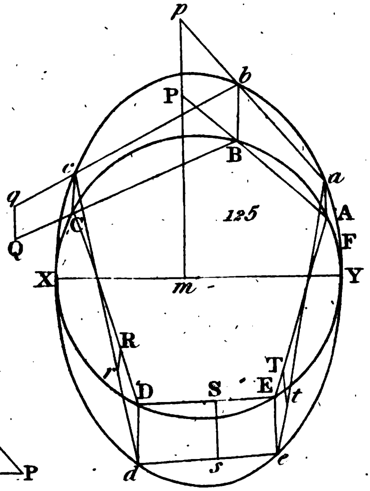
SOLUTION, *by* ASTRONOMICUS.

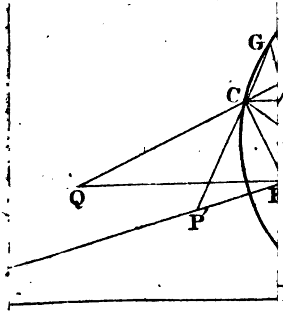
The greatest apparent distance of a star from the north pole, arising from aberration and nutation together, is when the sun's longitude, and also that of the moon's ascending node, are three signs *before* the star, reckoning according to the order of the signs; and that apparent distance will be greatest of all when the star is in the solstitial colure; but it does not appear, that the period can be accurately determined when such an event will take place, for any proposed star. We can, however, calculate the time which seems to be required by the question, if it falls within the limits of some proposed revolution of the moon's nodes, or if it is restricted to a given year: Thus, for example, let the year be 1809, and suppose the star's right ascension = 108°, and its declination = 60° north. Let a great circle be drawn, (to the eastward) from the star, perpendicular to the hour circle or meridian passing through the star, and it will meet the ecliptic in 25° 41′ of *libra*, which is the sun's place, when the aberration in declination southward is a *maximum*, (the obliquity of the ecliptic being 23° 27′ 44″,) this answers to Oct. 19, at London; on that day the star comes to the meridian at 22½ *min.* before 6 in the morning. The longitude of the moon's node at that time is 6*s*. 23° 45′, and that of the sun 6*s*. 25° 26′; the star therefore, is very nearly at its maximum distance from the north pole, which distance will not sensibly vary during 3 or 4 days at that time.

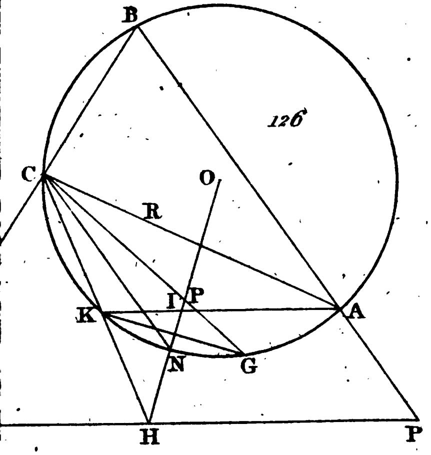
Vol.II. Plate VI. Fig. 117. to 128.

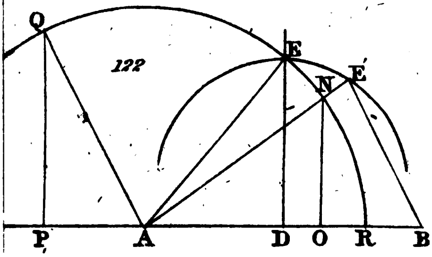


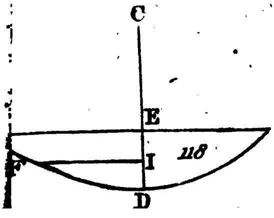












NOTICES

RELATING TO MATHEMATICS.

I. MATHEMATICIANS LATELY DECEASED.

On the 7th of April, 1807, M. D. Lalande, the celebrated French Astronomer, aged 75. He was the Author of many Memoirs connected with his favourite Science. His most valuable and best known work, however, is his *Astronomie* in 3 vols. Quarto, of which the third edition was published in 1792.

In July, 1807, George Atwood, Esq. M. A. and F. R. S. aged 62. He was for some time Tutor, and many years Fellow of Trinity College, Cambridge; and was highly distinguished for his Mathematical Acquirements. In 1784, he published in one volume, octavo, "A Treatise on the Rectilinear Motion, and Rotation of Bodies, with a description of original Experiments relative to the subject." At the same time he published, "An Analysis of a Course of Lectures, on the Principles of Natural Philosophy, read in the University of Cambridge." In 1801, he published a "Dissertation on the Construction and Properties of Arches," and in the course of that year, he also published a Supplement to it. Mr. Atwood also, at different times, contributed several valuable Memoirs to the Transactions of the Royal Society of London.

II. ANOTHER NEW PLANET.

On the 29th of March, 1807, Dr. Olbers, at Bremen, discovered another new planet, to which he has given the name of Vesta, being the second which we owe to the observations of this learned and indefatigable astronomer. It resembles a star of the sixth magnitude, and is very like in its appearance to the Georgium Sidus. This planet is found in the same region between Mars and Jupiter, in which Ceres, Pallas, and Juno, perform their revolutions round the Sun. Its place, as observed by Mr. Firminger at the Royal Observatory, Greenwich, on the 22d of June, 1807, 10^h. 29^m. 48^s. mean time was

App. A. R.	App.Dec.N.	Longitude.	Lat N.
$6^s. 4°. 23'. 30''$ —	$6°. 36'. 3''$ —	$6^s. 1°. 23'. 18''$ —	$7°. 47'. 0''$.

III. TRANSACTIONS OF THE ROYAL SOCIETIES OF LONDON AND EDINBURGH.

We propose, in future, to give the titles of the papers relating to Mathematics, contained in the Transactions of the Royal Society of London, as they appear. And we shall, at present, enumerate all those which have been published in that work, since the commencement of the present century.

The Vol. for 1800 contains, 1. On the method of determining from the real probabilities of life, the nature of contingent reversions, in which three lives are involved in the survivorship. By Wm. Morgan, Esq. F. R. S. 2. On the power of penetrating into space by telescopes, &c. By Dr. Herschell. 3. A second appendix to the improved solution of a problem in physical astronomy, inserted in the philosophical transactions for 1798. By the Rev. John Hellins, B. D. F. R. S. 4. Outlines of experiments and inquiries respecting sound. By Thomas Young, M. D. F. R. S. 5. On double images, caused by atmospherical refraction. By Wm. Hyde, Wollaston, M. D. F. R S. 6. Investigation of the powers of the prismatic colours to heat and illuminate objects: with remarks that prove the different refrangibility of radient heat. By Dr. Herschell. 7. Experiments on the refrangibility of the invisible rays of the sun. By Dr. Herschell. 8. Ex-

By Dr. Herschell. 9. An account of the trigonometrical survey, carried on in the years 1797, 1798, and 1799. By Captain Wm. Mudge, of the Royal Artillery.

The Vol. for 1801 contains, 1. The Bakerian Lecture. On the mechanism of the eye. By Dr. Young. 2. On the necessary truth of certain conclusions, obtained by means of imaginary quantities. By Robert Woodhouse, A. M. 3. Demonstration of a theorem by which such portions of the solidity of a sphere are assigned as admit of an algebraic expression. By Mr. Woodhouse. 4. Observations tending to investigate the nature of the sun, &c. By Dr. Herschell. 5. On an improved reflecting circle. By Joseph de Mendoza Rios, Esq. F. R. S.

The Vol. for 1802 contains, 1. The Bakerian Lecture. On the theory of light and colours. By Dr. Young. 2. On the independence of the analytical and geometrical methods of investigation, and on the advantages to be derived from their separation. By Mr. Woodhouse. 3. Observations on the two lately discovered celestial bodies. By Dr. Herschell. 4. A method of examining refracting and dispersive powers, by prismatic reflection. By Dr. Wollaston. 5. On the oblique refraction of Iceland crystal. By Dr. Wollaston. 6. An account of some cases of the production of colours not hitherto described. By Dr. Young. 7. Of the rectification of the conic sections. By the Rev. Mr. Hellins.

The Vol. for 1803 contains, 1. The Bakerian Lecture. Observations on the quantity of horizontal refraction; with a method of measuring the dip at sea. By Dr. Wollaston. 2. Observations on the transit of Mercury over the disk of the sun, &c. By Dr. Herschell. 3. Account of the changes that have happened during the last twenty-five years, in the relative situation of double stars, &c. By Dr. Herschell. 4. Account of the measurement of an arc of the meridian, extending from Dunnose, in the Isle of Wight, to Clifton, in Yorkshire, in course of the operations carried on for the trigonometrical survey of England in the years 1800, 1801, and 1802. By Major Mudge.

The Vol. for 1804 contains, 1. The Bakerian Lecture. Experiments and calculations relative to physical optics. By Dr. Young. 2. On the integration of certain differential expressions, with which problems in physical astronomy are connected, &c. By Mr. Woodhouse. 3. Continuation of an account of the changes that have happened in the relative situation of double stars. By Dr. Herschell.

The Vol. for 1805 contains, 1. Experiments for ascertaining how far telescopes will enable us to determine very small angles, &c. with an application of the result of these experiments to a series of observations on the nature and magnitude of Mr. Harding's lately discovered star. By Dr. Herschell. 2. An essay on the cohesion of fluids. By Dr. Young. 3. On the direction and velocity of the motion of the sun and solar system. By Dr. Herschell. 4. Observations on the singular figure of the planet Saturn. By Dr. Herschell.

The Vol. for 1806 contains, 1. The Bakerian Lecture. On the force of percussion. By Dr. Wollaston. 2. Mémoire sur les quantités imaginare. Par M. Buée. 3. The application of a method of differences to the species of series whose sums are obtained by Mr. Landen, by the help of impossible quantities. By Mr. Benj. Gompertz. 4. On the quantity and velocity of the solar motion. By Dr. Herschell. 5. A new demonstration of the binomial theorem, when the exponent is a positive or negative fraction. By the Rev. Abram Robertson, A. M. F. R. S. Savilian Professor of Geometry in the University of Oxford. 6. New method of computing logarithms. By Thomas Manning, Esq. 7. On the declination of some of the principal fixed stars: with a description of an astronomical circle. By John Pond, Esq. 8. Observations and remarks on the figure, the climate, and the atmosphere, of Saturn and its ring. By Dr. Herschell.

The Vol. for 1807 contains, 1. On the precession of the equinoxes. By the Rev. Abram Robertson. 2. An investigation of the general term of an important series in the inverse method of finite differences. By the Rev. John Brinkley, D. D. F. R. S. and Andrew's Professor of Astronomy in the University of Dublin. 3. Experiments for investigating the cause of the coloured concentric rings, discovered by Sir Isaac Newton, between two object glasses laid one upon another. By Dr. Herschell. 4. Observations and measurements of the planet Vesta. By Mr. Schroeter. 5. Observations on the nature of the new celestial body, discovered by Dr. Olbers, and of the comet which was expected to appear last January, on its return from the sun. By Dr. Herschell.

The first part of the sixth vol. of the Transactions of the Royal Society of Edinburgh is now published. As it, perhaps, may be agreeable to our readers to have a list of the memoirs relating to mathematical subjects contained in this work from its commencement, we shall enumerate them as follows.

VOL. I. contains, 1. Biographical account of Dr. Matthew Stewart. By Mr. Professor Playfair. 2. On the causes which affect the accuracy of barometrical measurements. By Mr. Playfair. 3. On the use of negative quantities in the solution of problems. By Mr. Wm. Greenfield. 4. An improvement of the method of correcting the observed distance of the moon from the sun or from a fixed star. By the Rev. Thomas Elliott. 5. Orbit and motion of the Georgium Sidus, determined directly from observations. By Mr. John Robison, Professor of Natural Philosophy in the University of Edinburgh.

VOL. II. contains, 1. Abstract of experiments made to determine the true resistance of the air to the surfaces of bodies. By Dr. Charles Hutton, of the Royal Military Academy, Woolwich. 2. Observations on the places of the Georgium Sidus, made at Edinburgh with an equatorial instrument. By Professor Robison. 3. On the motion of light, as affected by refracting and reflecting substances, which are also in motion. By Professor Robison. 4. Demonstrations of some of Dr. Stewart's general theorems. By the Rev. Dr. Small. 5. Remarks on the astronomy of the Brahmins. By Mr. Playfair. 6. On the resolution of indeterminate problems. By Mr. John Leslie.

VOL. III. contains, 1. Experiments and observations on the unequal refrangibility of light. By Dr. Robert Blair, Professor of Practical Astronomy in the University of Edinburgh. 2. On the origin and investigation of Porisms. By Mr. Playfair.

VOL. IV. contains, 1. Four theorems for resolving all the cases of plane and spherical triangles. By the Rev. Mr. Fisher. 2. On the principles of the antecedental calculus. By James Glenie, Esq. 3. Observations on the trigonometrical tables of the Brahmins. By Mr. Playfair. 4. Some geometrical porisms, with examples of their application to the solution of prolongitude of Aberdeen. By Dr. Andrew Mackay. 6. A new series for the rectification of the ellipsis, with observations of the evolution of the formula $(a^2 + b^2 - 2ab \cos. \phi)^n$. By James Ivory, A. M.

VOL. V. contains, 1. Investigation of certain theorems relating to the figure of the earth. By Mr. Playfair. 2. New method of resolving cubic equations. By Mr. Ivory. 3. A new and universal solution of Kepler's problem. By Mr. Ivory. 4. A new method of expressing the coefficients of the development of the algebraic formula $(a^2 + b^2 - 2ab \cos. \phi)^n$, by means of the perimeters of two ellipses; with an appendix, containing an investigation of a formula for the rectification of any arch of an ellipse. By Mr. Wallace. 5. Rule for reducing a square root to a continued fraction. By Mr. Ivory.

VOL. VI. PART I. contains a geometrical investigation of some curious and interesting properties of the circle, &c. By Mr. Glenie.

IV. MATHEMATICAL WOKKS LATELY PUBLISHED.

Scriptores Logarithmici, Vol. VI. 4to. By Francis Maseres, Esq. F. R. S. *Cursitor Baron of the Exchequer.*

An Elementary Treatise on Natural Philosophy. Translated from the French of M. R. I. Hauy, Professor of Mineralogy at the Museum of Natural History, &c. By Olinthus Gregory, A. M. *of the Royal Military Academy, Woolwich. 2 Vols, 8vo. with plates. A second Edition of the original French Work has also been lately published, and* Mr. Gregory *has availed himself of it in his Translation.*

A Reply to a critical and monthly Reviewer, in which is inserted Euler's Demonstration of the Binomial Theorem. By Abram Robertson, D. D. F. R. S. *Savilian Professor of Geometry.*

FOREIGN BOOKS.

Journal De L'Ecloe Polytechnique, 13th Cahier. The 9th and 10th Cahiers, containing the remainder of the Mechanique Philosophique, by Prony; *and the 14th Cahier are in the Press.*

Application De L'Analyse A La Geometrie, par Monge *and* Hachette, 2 *Parts in Quarto.*

Correspondence sur Le Ecole Polytechnique, 6 *Numeros in Octavo.*

Essai De Geometrie Analytique appliqué aux Courbes et aux Surfaces du Second Ordre. Par I. B. Biot. *Second Edition.*

A French Translation of the Disquisitiones Arithmeticæ *of* Mr. Gaus *has appeared. It is in 1 Vol. 4to. and is called* Recherches Arithmetiques. *Our readers will recollect that we have already noticed the original work at page 77 of Vol. I of the Mathematical Repository, where we have given his formula for the calculation of the side of a regular polygon of 17 sides, inscribed in a circle.*

Nouvelle Méthode pour la Résolution des Équations Numériques d'un degré quelconque; D'apres laquelle tout le calcul exigé pour cette Resólution se reduit à lémploi des deux premiéres régles de l'Arithmetique. Par F. D. Budan.

Mémoire sur la relation qui existe entres les distances de cinq

ARTICLE IV.

Solutions to Questions proposed in Number VII.

I. QUESTION 211, *from the* BIJA GONETA, *or* HINDOO ALGEBRA.

Find two integer ſquare numbers whoſe ſum and difference diminiſhed by 1 ſhall both be ſquares?

"The *Bija Goneta* was written in Sanscrit by BHASKER ACHARIJ, a famous *Hindoo* Mathematician and Astronomer, who lived about the beginning of the 13th century of the *Christian* era: and was translated into Persian in 1634."

FIRST SOLUTION, *from the* BIJA GONETA, *or* HINDOO ALGEBRA.

Assume $4x^2$ and $5x^2+1$ for the two squares:
Then $4x^2+5x^2+1-1=9x^2$ a square: And $5x^2+1-4x^2-1=x^2$ a square;
Therefore we have to make $5x^2+1$ a square: Now if we assume $x=4$, then $5\times16+1=81$, and $4\times16=64$, are the two squares: For $81+64-1=144$ a square; and $81-64-1=16$ a square. And if $x=72$, the two squares will be 25921 and 20736.

We shall subjoin the three following Questions, with the methods of solution from the same work, (the *Bija Goneta.*)

1. What two integer numbers are those (5 and 6 excepted,) that when 3 is added to twice the difference of their squares, and also to 3 times that difference, the two results shall be square numbers?

Let $x=$ the difference of the squares. Then $2x+3=y^2$ (a square);
And $3x+3=z^2$ (a square):

Since $2x+3=y^2$, $x=\frac{y^2-3}{2}$, and $3x+3=\frac{3y^2-3}{2}=z^2$,

whence $9y^2=6z^2+9$, and $3y=(6z^2+9)^{\frac{1}{2}}$. If z be assumed $=6$, then $3y=15$, and $y=5$, and $x=11$, and the two numbers are 5 and 6, which, by the question, are excepted.
Now let $z=60$; then $3y=147$, and $y=49$, whence

$x = 1199$ the difference of the two squares. Let m and n be the required numbers, or roots of the squares; then $(m + n)(m - n) = 1199$: assume $m - n = 1$; then $2m = 1200$, and $m = 600$ one of the required numbers, and $n = 599$ the other.

2. To find x and y in whole numbers, when $x + y$ and $x^2 + y^3$ are both squares?

Assume $\frac{y^2 - y}{2} = x$, then $\left(\frac{y^2 - y}{2}\right)^2 + y^3 = \frac{y^4 + 2y^3 + y^2}{4}$ a square, its root being $\frac{y^2 + y}{2}$. And since $x = \frac{y^2 - y}{2}$, $x + y$ is $= \frac{y^2 + y}{2}$: Let $\frac{y^2 + y}{2} = a^2$, then $y^2 + y = 2a^2$, or $4y^2 + 4y = 8a^2$, and $4y^2 + 4y + 1 = 8a^2 + 1$, whence $2y + 1 = (8a^2 + 1)^{\frac{1}{2}}$. If $a = 6$, then $y = 8$, and $x = 28$, which numbers answer the question. And if $a = 35$, we get 49 and 1176 for the two numbers.

3. To find x and y, when $x^2 + xy + y^2$ and $xy(x + y) + 1$ are both integer squares?

First, $(x^2 + xy + y^2) \times 36 \ldots\ldots = 36x^2 + 36xy + 36y^2$

Assume the square $(6x + 3y)^2 \ldots\ldots = 36x^2 + 36xy + 9y^2$

difference $\ldots\ldots\ldots\ldots 27y^2$

Now $\frac{27y^2}{y} = 27y$, and $27y - y = 26y$, and $\frac{26y}{2} = 13y$. Make the root

$6x + 3y = 13y$, then $x = \frac{5y}{3}$, $x + y = \frac{8y}{3}$, and $xy = \frac{5y^2}{3}$:

then $x^2 + xy + y^2 = \frac{25y^2}{9} + \frac{5y^2}{3} + y^2 = \frac{49y^2}{9}$, and its root $= \frac{7y}{3}$.

Next, $\frac{7y}{3} \times \frac{8y}{3} = \frac{56y^2}{9}$, and $\frac{56y^2}{9} + 1$ or $\frac{56y^2 + 9}{9}$ is to be a square: Put $\frac{56y^2 + 9}{9} = a^2$, then $(56y^2 + 9)^{\frac{1}{2}} = 3a$. Now if $y = 6$, then $a = 15$, whence $x = 10$; therefore, 10 and 6 are the values of x and y.

Remark by the transcriber. The method of making $x^2 + xy + y^2$ a rational square, appears not to be general; for any square, (instead of 36,) will not answer the purpose. Nor is there any obvious step in the process which leads to the reason for adopting the root $\frac{7y}{3}$ instead of $\frac{5y^2}{3}$ (or xy) in rendering the expression $xy(x+y)+1$ a rational square: For the same value of y will result if we use $\frac{5y^2}{3}$; thus let $\frac{5y^2}{3}\times\frac{8y}{3}+1=b^2$ (a square); then $40y^3+9=9b^2$; and the expression $40y^3+9$ is a square when $y=1$, or 6.

SECOND SOLUTION, *by Mr.* CUNLIFFE, *R. M. College.*

Let $a^2+2ab+b^2$ and a^2 denote the two squares: Then by the question

$$2a^2+2ab+b^2-1=c^2,$$

and

$$2ab+b^2-1=d^2.$$

The sum and difference of these equations, give

$$2a^2+4ab+2b^2-2=c^2+d^2,$$

and

$$2a^2=c^2-d^2:$$

Whence $a^2+2ab+b^2=(a+b)^2=\frac{1}{2}(c^2+d^2)+1$, and $a^2=\frac{1}{2}(c^2-d^2)$.

Put $d+n=c$; then $a^2=\frac{1}{2}(c^2-d^2)=\frac{1}{2}n^2+dn=r^2n^2$; whence $n=\frac{2d}{2r^2-1}$, $c=d+n=d+\frac{2d}{2r^2-1}=\frac{d(2r^2+1)}{2r^2-1}$, and by means of these $1+\frac{c^2+d^2}{2}=1+\frac{d^2(4r^4+1)}{2r^2-1}$,

Put $d=s(2r^2-1)$; then $1+\frac{d^2(4r^4+1)}{2r^2-1}=1+s^2(4r^4+1)$ $=$ a square $=(ts+1)^2=t^2s^2+2ts+1$: whence $s=\frac{2t}{4r^4+1-t^2}$ where r and t must be so related as to give s a whole number.

Take $t=2r^2$, then $s=4r^2$, $d=s(2r^2-1)=4r^2(2r^2-1)$, $n=2d\div(2r^2-1)=8r^2$, $c=d+n=4r^2(2r^2+1)$, $a^2=\frac{1}{2}(c^2-d^2)=64r^6$, and $(a+b)^2=\frac{1}{2}(c^2+d^2+2)=64r^8+16r^4+1$, which are the two required squares, where r may be taken at pleasure.

Take $r=1$, then the two squares are 81 and 64.

Take $r=2$, and the two squares are 16641 and 4096.

This question was also answered by Dr. O'Riordan.

II. QUESTION 212, *by* INVESTIGATOR.

What ſum ſhould A, aged 30, pay on the event of his ſurviving B, aged 65, for an annuity of 1000*l.* during their joint lives, uſing the Northampton table of the probable duration of life, and allowing 5 per cent. intereſt of money?

SOLUTION, *by* INVESTIGATOR, *the Proposer.*

As this question contains in fact two distinct questions, viz. 1st. To find what the present value is of an annuity of £1000, to continue during the joint lives of A and B; and 2dly. What sum is equivalent to that present value, if deferred till the death of B, and limited also to the restriction of A's being then alive; so the complete solution may naturally be divided into two parts, each of which we shall consider in order.

1. Taking a, b, c, d, e, &c. to denote the number of the living at the age of A, and at the end of the 1st, 2d, 3d, 4th, &c. succeeding year; and p, q, r, s, t, &c. the number of the living corresponding to the age of B, to B's age $+ 1$, $+ 2$, $+ 3$, $+ 4$, &c, respectively, as found in Dr. Price's Northampton Table of the probable duration of human life, and putting R for the amount of £1 in a year, (i. e. 1·05 when it is 5 per cent. interest,) we shall have

The probability that

	A will be alive at the end of the	B will be alive at the end of the	And that both A and B will be alive at the end of the
1st Year	$= \frac{b}{a}$ ……	$= \frac{q}{p}$ …………	$= \frac{bq}{ap}$
2d ……	$= \frac{c}{a}$ ……	$= \frac{r}{p}$ …………	$= \frac{cr}{ap}$
3d ……	$= \frac{d}{a}$ ……	$= \frac{s}{p}$ …………	$= \frac{ds}{ap}$
4th ……	$= \frac{e}{a}$ ……	$= \frac{t}{p}$ …………	$= \frac{et}{ap}$
&c ……	&c. …………	&c. ……………	&c.

Therefore, by discounting the first term of the last rank of probabilities, namely, $\frac{bq}{ap}$, for one year, (i. e. multiplying it by R^{-1}); the second, $\frac{cr}{ap}$, for 2 years, (i. e. multiplying it by R^{-2}); the third, $\frac{ds}{ap}$, for 3 years, (i. e. multiplying it by R^{-3}); the fourth $\frac{et}{ap}$, for 4 years, (i. e. multiplying it by R^{-4}), &c. and collecting the several products, and multiplying the whole sum by 1000, the present value of an annuity of £1000 upon the joint lives of A and B, is found to be equal to

$$£.\ 1000\left(\frac{bq}{apR}+\frac{cr}{apR^2}+\frac{ds}{apR^3}+\frac{et}{apR^4}+\text{\&c. to the end}\right.$$

of the table, or till one of the factors in the numerator becomes $= 0$,) or equal to

$\frac{1000}{apR}\left(bq+\frac{cr}{R}+\frac{ds}{R^2}+\frac{et}{R^3}+\text{\&c.}\right)$ from which expression, the value is readily computed, and will be found to be $=$ £6447.841.

2. Let the value of the annuity just found, (£6447.841,) be put $= P$, the equivalent contingent sum, that is to be paid in case A survives B, $= Q$, and let the same symbols as in the preceding investigation be retained, and we shall have the probability of

A's being alive at the end of the *first* year.......... $= \frac{b}{a}$

A's dying in the first year........................ $= \frac{a-b}{a}$

B's dying in the first year........................ $= \frac{p-q}{p}$

A's surviving B in, and being alive at the end of the 1st year $= \frac{b}{a}\left(\frac{p-q}{p}\right)$

and the probability of their both dying in that year, and B dying first, nearly*, $= \frac{1}{2}\left(\frac{a-b}{a}\right)\left(\frac{p-q}{p}\right)$.

Again, the probability of

A's being alive at the end of the second year, is...... $= \frac{c}{a}$

A's dying in the second year...................... $= \frac{b-c}{a}$

B's dying in the second year...................... $= \frac{q-r}{p}$

A's surviving B in, & being alive at the end of the 2d year $= \frac{c}{a}\left(\frac{q-r}{p}\right)$

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* I say nearly, for although our best writers on the subject have, as it were consentaneously, taken this as the *exact* chance that is above mentioned, yet, of its differing from the truth in many cases, when the period of time is so long as a year, there cannot be entertained a moment's doubt.

and the probability of their both dying in that year, and B dying first, nearly, $= \frac{1}{2}(\frac{b-c}{a})(\frac{q-r}{p})$.

In like manner, the probability of

A's being alive at the end of the third year is .... $= \frac{d}{a}$

A's dying in the third year .... $= \frac{c-d}{a}$

B's dying in the third year .... $= \frac{r-s}{p}$

A's surviving B in, & being alive at the end of the 3d year $= \frac{d}{a}(\frac{r-s}{p})$

and the probability of their both dying in the 3d. year, and B dying first, nearly, $= \frac{1}{2}(\frac{c-d}{a})(\frac{r-s}{p})$.

And also, the probability of

A's being alive at the end of the fourth year is .... $= \frac{e}{a}$

A's dying in the fourth year .... $= \frac{d-e}{a}$

B's dying in the fourth year .... $= \frac{s-t}{p}$

A's surviving B in, & being alive at the end of the 4th year $= \frac{e}{a}(\frac{s-t}{p})$

and the probability of their both dying in the fourth year, and B dying first, nearly, $= \frac{1}{2}(\frac{d-e}{a})(\frac{s-t}{p})$.

&c. &c.

Therefore, taking it as an equality of chance, that the event of A's surviving B, may happen in the *first* as in the *second half* of any one year, and allowing the proper discount on the two last mentioned probabilities in each of the above mentioned years, respectively, and collecting the several quantities, there will arise the following equation, viz. P = Q ×

$$\left\{\begin{array}{l} \frac{b}{aR^{\frac{1}{2}}}(\frac{p-q}{p}) + \frac{c}{aR^{\frac{3}{2}}}(\frac{q-r}{p}) + \frac{d}{aR^{\frac{5}{2}}}(\frac{r-s}{p}) + \frac{e}{aR^{\frac{7}{2}}}(\frac{s-t}{p}) + \text{\&c.} \\ + \frac{a-b}{2aR^{\frac{1}{2}}}(\frac{p-q}{p}) + \frac{b-c}{2aR^{\frac{3}{2}}}(\frac{q-r}{p}) + \frac{c-d}{2aR^{\frac{5}{2}}}(\frac{r-s}{p}) + \frac{d-e}{2aR^{\frac{7}{2}}}(\frac{s-t}{p}) + \text{\&c.} \end{array}\right\}$$

Or, by taking the sum of the two series, and dividing all by Q,

$$\frac{P}{Q}\text{ is }=\frac{(a+b)(p-q)}{2apR^{\frac{1}{2}}}+\frac{(b+c)(q-r)}{2apR^{\frac{3}{2}}}+\frac{(c+d)(r-s)}{2apR^{\frac{4}{2}}}+\frac{(d+e)(s-t)}{2apR^{\frac{7}{2}}}+\&c.$$

And, if we multiply the whole by $2apR^{\frac{1}{2}}$,

$$\frac{2PapR^{\frac{1}{2}}}{Q}=(a+b)(p-q)+\frac{(b+c)(q-r)}{R}+\frac{(c+d)(r-s)}{R^2}+\frac{(d+e)(s-t)}{R^3}+\&c.$$

and, consequently,

$$Q=2PapR^{\frac{1}{2}}\div\left((a+b)(p-q)+\frac{(b+c)(q-r)}{R}+\frac{(c+d)(r-s)}{R^2}+\frac{(d+e)(s-t)}{R^2}\right.$$

$+\ \&c.)=$ £12224·69, or £12224. 13s. 9½*d.* the sum that ought to be stipulated to be paid by A, at the death of B, for the annuity mentioned in the question.

Remark 1. If we would avail ourselves of the assistanee of the tables of annuities upon joint lives, the series that expresses the value of $\frac{P}{Q}$, when the multiplications are performed in the numerators of the several fractions of which it is composed, becomes equal to

$$\frac{1}{2apR^{\frac{1}{2}}}\times\left(ap+bp-aq-bq+\frac{bq+cq-br-cr}{R}+\frac{cr+dr-cs-ds}{R^2}+\frac{ds+es-dt-et}{R^3}+\&c.\right)$$

Which series is evidently equal to the four following, viz.

$$\frac{1}{2apR^{\frac{1}{2}}}\times\left\{\begin{array}{l} \left(ap+\frac{bq}{R}+\frac{cr}{R^2}+\frac{ds}{R^3}+\&c.\right. \\ \left(bp+\frac{cq}{R}+\frac{dr}{R^2}+\frac{es}{R^3}+\&c.\right. \\ -\left(aq+\frac{br}{R}+\frac{cs}{R^2}+\frac{dt}{R^3}+\&c.\right. \\ -\left(bq+\frac{cr}{R}+\frac{ds}{R^2}+\frac{et}{R^3}+\&c.\right. \end{array}\right.$$

Let α and φ denote the number of the living corresponding to 'A and 'B, two lives whose ages are less by one year than the ages of A and B, and let 'A'B, A'B, 'AB and AB denote the value of the joint lives of 'A and 'B, A and 'B, 'A and B, and A and B respectively, as taken from the tables of annuities upon joint lives; then, by multiplying the first of the foregoing series by $\frac{ap}{\alpha\varphi R^{\frac{1}{2}}}$, the second by $\frac{p}{\varphi R^{\frac{1}{2}}}$, the third by $\frac{a}{\alpha R^{\frac{1}{2}}}$, and the fourth by $\bar{R}^{\frac{1}{2}}$, there will be produced

$$\left.\begin{array}{l} \frac{1}{2a\varphi R} \times \left(ap + \frac{bq}{R} + \frac{cr}{R^2} + \frac{ds}{R^3} + \&c.\right. \\ \frac{1}{2a\varphi R} \times \left(bq + \frac{cq}{R} + \frac{dr}{R^2} + \frac{es}{R^3} + \&c.\right. \end{array}\right] . \qquad \left.\begin{array}{l} -\frac{1}{2apR} \times \left(aq + \frac{br}{R} + \frac{cs}{R^2} + \frac{dt}{R^3} + \&c.\right. \\ -\frac{1}{2apR} \times \left(bq + \frac{cr}{R} + \frac{ds}{R^2} + \frac{et}{R^3} + \&c.\right. \end{array}\right\}$$

Of which expressions the first is evidently $= \frac{1}{2}$'A'B, the second $= \frac{1}{2}$A'B, the third $= -\frac{1}{2}$A'B, and the fourth $= -\frac{1}{2}$AB. Therefore, by dividing by the preceding factors, &c. Q is found to be eqnal $P + \frac{R^{\frac{1}{2}}}{2}(\text{'A'B} \times \frac{a\varphi}{ap} + \text{A'B} \times \frac{\varphi}{p} - \text{'AB} \times \frac{a}{a} - \text{AB})$; which is a very commodious theorem for obtaining the value of Q by means of the tables of annuities upon joint lives and the original table of the probable duration of human life.

Remark 2. The value of Q, if computed by Mr. Simpson's Approximation, aided by Dr. Price's correction, (see Dr. P—'s Revers. Payments, vol. 1, pa. 37 and 38, 6th Edit.) comes out = £12702, exceeding the true value by £477. 6s. 2d. If computed by the rule on pa. 40 and 41 of the same vol. it will be found to be £12535, exceeding the true value by £310. 6s. 2d. If computed directly from the Northampton table, and discount be allowed on the several probabilities up to the end of each year respectively, (the common, though erroneous method of allowing discount,) it will be found = £12525, differing from the true value by £300. 6s. 2d. A certain ingenious calculator makes it £13059, differing from the true value by above £834!

Rem. 3. Were we in possession of tables of the probable duration of human life for much shorter periods of time than years, and complete tables of annuities answering thereto, it would be easy to deduce a more exact value than even that which is exhibited by the above theorem, something similar to the ingenious method made use of in Art. xi, No. 7, of the New Series of the Repository, but considering the vague nature of the subject, and the tables now in common use, it is presumed, that the exactness of the above method of solution is abundantly sufficient for all practical purposes, and the ease with which the value of Q may be computed by means of the theorem in Rem. 1, cannot fail of recommending it in preference to some other *less correct* and *more troublesome* rules that have heretofore *been given*.

This question was also answered by Mr, J. H, Hearding, *of the Globe Insurance Office.*

III. QUESTION 213, *by Mr.* BAZLEY.

Find a circle whereof the diameter, and the chord, ſine, and verſed-ſine of an arc of it, are all integers?

SOLUTION, *by Mr.* CUNLIFFE.

Let sz denote the sine, and vz the versed sine of an arc; then will $z\sqrt{(v^2+s^2)}$ denote the chord, and $z(v^2+s^2)\div v$ the diameter of the circle, which will all be rational, when v^2+s^2 is a rational square. Put $v = 2mn$, and $s = m^2-n^2$; then $z\sqrt{(v^2+s^2)} = z(m^2+n^2)$, and $z(v^2+s^2)\div v = z(m^2+n^2)\div 2mn$.

Put $z = 2mn$, then sine $sz = 2mn(m^2-n^2)$; versed sine $vz = 4m^2n^2$; chord $z\sqrt{(v^2+s^2)} = 2mn(m^2+n^2)$, and diameter $z(v^2+s^2)\div v = (m^2+n^2)^2$, where m and n denote any whole numbers.

The preceding expressions give only one arc that will answer to the same radius; but any number of arcs of the same circle whose chords, sines, and versed sines will be rational, may be found as follows:

Put $m^2+n^2 = m'^2+n'^2 = m''^2+n''^2$, &c. then by the known method of dividing the sum of two squares into two others, we may put

$$m' = \frac{m(p^2-q^2)+2npq}{p^2+q^2} \text{ and } n' = \frac{n(p^2-q^2)-2mpq}{p^2+q^2},$$

$$m'' = \frac{m(p'^2-q'^2)+2np'q'}{p'^2+q'^2} \text{ and } n'' = \frac{n(p'^2-q'^2)-2mp'q'}{p'^2+q'^2}, \text{ \&c.}$$

where supposing m and n given, p, q; p', q', &c. may be taken at pleasure, but not in the same ratio. Then the diameter will be expressed by $(m^2+n^2)^2$; the chords by $2mn(m^2+n^2)$, $2m'n'(m^2+n^2)$, $2m''n''(m^2+n^2)$, &c.; the sines by $2mn(m^2-n^2)$, $2m'n'(m'^2-n'^2)$, $2m''n''(m''^2-n''^2)$, &c. and the versed sines by $4m^2n^2$, $4m'^2n'^2$, $4m''^2n''^2$ &c.

This question was also answered by Dr. O'Riordan,

IV. QUESTION 214, *by* AMICUS.

The ends A, C of two given beams are moveable about two fixed points, and the other ends B, D are connected by a ſtring of a given length which paſſes over a fixed pully. It is required to determine the poſition of the beams when they are in equilibrio; the pully and beams being in the ſame vertical plane?

SOLUTION, *by Mr.* J. JOHNSON, *Birmingham.*

Let P, (fig. 129, pl. 7,) be the point where the pully is fixed, and draw AE, BG, CF, DH perpendicular to the horizontal line EPF. Also, PK, PL parallel to the beams AB, CD, meeting the lines BG, DH, produced, in K and L; and let Q, R, be the centres of gravity of the beams AB, CD; and W, w their respective weights. Then, by the property of the lever, the force which, acting in the direction BG, would sustain the end B, is to the whole weight of the beam, as AQ to AB, and is therefore equal to (AQ ÷ AB) W; which being resolved into a force acting in the direction of the string BP, is = (BP ÷ BK) (AQ ÷ AB) W. In like manner, the force acting in the direction of the string DP is = (DP ÷ DL) (RC ÷ DC) w; and when the beams are in equilibrio, these forces must be equal; that is,

$$\frac{BP}{BK} \times \frac{AQ}{AB} \times W = \frac{DP}{DL} \times \frac{RC}{DC} \times w.$$

Put EP $= a$, AE $= b$, PF $= c$, CF $= d$, AB $= m$, AQ $= n$, DC $= r$, RC $= s$, GB $= x$, PG $= y$, PH $= w$, HD $= z$, and $l =$ the length of the string. Then BP $= \sqrt{(x^2+y^2)}$, GK $= y\,(x-b) \div (a-y)$, and BK $= x + y\,(x-b) \div (a-y) = (ax-by) \div (a-y)$; therefore

$$\frac{BP}{BK} \times \frac{AQ}{AB} \times W \text{ is } = \frac{a-y}{ax-by} \times \sqrt{(x^2+y^2)} \times \frac{n}{m} W.$$

And, in the same way we find

$$\frac{DP}{DL} \times \frac{RC}{DC} \times w = \frac{c-w}{cz-dw} \times \sqrt{(w^2+z^2)} \times \frac{s}{r} w.$$

Therefore,

$$\frac{n}{m} W . \frac{a-y}{ax-by} . \sqrt{(x^2+y^2)} = \frac{s}{r} w . \frac{c-w}{cz-dw} . \sqrt{(w^2+z^2)}.$$

From this equation, and the equations $\sqrt{(x^2+y^2)} + \sqrt{(w^2+z^2)} = l$, $(a-y)^2 - (c-b)^2 = m^2$, and $(c-w)^2 + (z-d)^2 = r^2$ the value of the unknown quantities may be determined.

Solutions were also received from Messrs. Amicus, Cunliffe, *and* O'Riordan.

V. QUESTION 215, *by* HYPATIA.

This question has, by some mistake, been wrong proposed; it ought to have been as follows:

From A, one end of AB, the diameter of a circle, draw any chord AC. Biſect the arch AC in E, and join BE, meeting AC in F, and draw ED perpendicular to AB meeting AC in G. The lines AG, GF ſhall be equal. Required the demonſtration?

SOLUTION, *by Dr.* O'RIORDAN.

Fig. 130, pl. 7. Because the arc AC is bisected in E, the angles CAE, EAB, AED are equal; therefore the right-angled triangles AEF, BDE, EDA are equiangular, or the angle GFE = to the angle GEF, and the angle GAE = to the angle GEA; therefore GF is = to GE, and AG = GE; consequently the lines AG, GF are equal.

Mr. J. Wallace, *teacher of Mathematics, at Edinburgh, sent a neat solution to this question.*

VI. QUESTION 216, *by Mr.* WALLACE, *R, M. College.*

Let a circle be given by poſition, and let A and B be two given points; a point C may be found, ſuch, that if any ſtraight line be drawn through it, meeting the circle in D and E, and AD, BD; alſo AE, BE be joined, there is a certain given ſpace which is a mean propoſtional between $AD^2 + BD^2$ and $AE^2 + BE^2$. Required the demonſtration?

SOLUTION, *by Mr.* LOWRY, *R. M. College.*

Fig. 131, pl. 7. Let the line joining the points A, B be bisected in P, and join PD, PE; then $AD^2 + BD^2 = 2AP^2 + 2PD^2$, and $AE^2 + BE^2 = 2AP^2 + 2PE^2$. Draw a line through P and the centre O, to meet the circle in H and K; and find the point V, so that $2OP \times OV$ may be equal to the square of the semi-diameter OH. Bisect PV in S, and draw DG, EN perpendicular to OH, then S is a given point; and by a well known theorem, $PD^2 = 2OP \times SG$, and $PE^2 = 2OP \times SN$.

Make $2OP \times SQ =$ to AP^2, then Q will also be a given point, and $AD^2 + BD^2 = 2AP^2 + 2PD^2 = 4OP \times QG$, and

$$AE^2 + BE^2 = 2AP^2 + 2PE^2 = 4OP \times QN,$$

Now

Now let $4OP \times l$ be the certain given space mentioned in the question, l being a line, the magnitude of which is yet to be determined;

$$\text{then } 4OP \times QG : 4OP \times l :: 4OP \times l : 4OP \times QN,$$
$$\text{or } QG : l :: l : QN;$$

therefore $QG \times QN$ is $= l^2 =$ to a given space, and therefore, the point C must be so situated, that, in whatever direction DE is drawn, the rectangle of the distances QG, QN shall be equal to a given space. This we know, (from the lemma, on page 125 of this vol.) will be the case when the point C is in the line OQ, and when $QC^2 = QH \times QK$ is $=$ the given space l^2, where the point C may be either within or without the circle. Hence this construction: Having found the point Q as in the analysis, take QC, on both sides of Q, a mean proportional between QH and QK; then either of the points C will answer the conditions of the proposition, and $4OP \times PC$ will be the given space.

The demonstration is evident from the analysis, and the lemma referred to above.

The same algebraically.

Bisect AB, (fig. 132, pl. 7.) in F, and join DF, EF. Draw FG to the centre G, and bisect FG in I: Also, draw DH, EK, CL perpendicular to FG. Put $FG = a$, radius $DG = r$, $AF = d$, $CL = b$, $GL = c$, the angle $DOC = \varphi$, $IH = x$, $IK = x'$, $GO = y$, and $p^2 =$ the given space.

Then $DF^2 - DG^2 = 2GF \times IH = 2ax$, or $DF^2 = 2ax + r^2$,
and $EF^2 - EG^2 = 2GF \times IK = 2ax'$, or $EF^2 = 2ax' + r^2$;
therefore $AD^2 + BD^2 = 2DF^2 + 2AF^2 = 4ax + 2r^2 + 2d^2$,
and $AE^2 + BE^2 = 2EF^2 + 2AF^2 = 4ax' + 2r^2 + 2d^2$;
or if $2r^2 + 2d^2$ be put $= 4ah$, we have $AD^2 + BD^2 = 4a(h+x)$ and $AE^2 + BE^2 = 4a(h + x')$; therefore by the question,

$$4a(h + x) : p^2 :: p^2 : 4a(h + x'),$$

$$\text{or,} \quad (h + x)(h + x') = \left(\frac{p^2}{4a}\right)^2.$$

Now by trigonometry, $OD = y \cos \varphi + \sqrt{(r^2 - y^2 \sin^2 \varphi)}$,
and $OE = -y \cos \varphi + \sqrt{(r^2 - y^2 \sin^2 \varphi)}$;
therefore $OH = y \cos^2 \varphi + \cos \varphi \sqrt{(r^2 - y^2 \sin^2 \varphi)}$,
and $OK = -y \cos^2 \varphi + \cos \varphi \sqrt{(r^2 - y^2 \sin^2 \varphi)}$.

But IO is $= \frac{1}{2}a - y$, therefore
$h + x$ is $= h + \frac{1}{2}a - y \sin^2 \varphi + \cos \varphi \sqrt{(r^2 - y^2 \sin^2 \varphi)}$, and
$h + x' = h + \frac{1}{2}a - y \sin^2 \varphi + \cos \varphi \sqrt{(r^2 - y^2 \sin^2 \varphi)}$, therefore,

$$(h+x)(h+x') = (h + \tfrac{1}{2}a - y \sin^2 \varphi)^2 - \cos^2 \varphi (r^2 - y^2 \sin^2 \varphi) = \left(\frac{p^2}{4a}\right)^2,$$

or. if m be put $= h + \frac{1}{2}a$, and $c + \frac{b \cos \varphi}{\sin \varphi} = GL + LO = z$, we have

$$m^2 - 2m \left(c + \frac{b \cos \varphi}{\sin \varphi} (\sin^2 \varphi - r^2 + r^2 \sin^2 \varphi + \left(c + \frac{b \cos \varphi}{\sin \varphi}\right)^2 \sin^2 \varphi = \left(\frac{p^2}{4a}\right)^2.$$

This equation contains all the conditions of the question; but since the angle φ is variable, we must determine what particular values must be assigned to the constant quantities, so that the equation may hold true whatever the value of φ may be. This will evidently be the case when $b = 0$, and $m^2 - r^2 = \left(\frac{p^2}{4a}\right)^2$; for then the equation becomes $-2mc \sin^2 \varphi + r^2 \sin^2 \varphi + c^2 \sin^2 \varphi = 0$, in which φ may be any angle we please. Dividing the last equation by $\sin^2 \varphi$, and completing the square, we have $c = m \pm \surd(m^2 - r^2)$.

The result of this investigation, then is, first, that $b = 0$, or that the point C is in the line GF, and coincides with the point L. Secondly, that $m^2 - r^2 = \left(\frac{p^2}{4a}\right)^2$, or $p^2 = 4a\surd(m^2 - r^2)$, the given space. And, thirdly, that c or the distance of the point C from the centre G, is $=$ to $m - \surd(m^2 - r^2)$, or to $m + \surd(m^2 - r^2)$, which shews that there are two points C that answer the conditions of the question; the one being within and the other without the circle.

Solutions were likewise received from Messrs. Cunliffe *and* O'Riordan,

VII. QUESTION 217, *by Mr.* LOWRY.

An ellipſe, and a ſtraight line, being given by poſition; a point may be found ſuch, that if any ſtraight line be drawn through that point to meet the ellipſe in two points, and perpendiculars be drawn from thoſe points to the ſtraight line given by poſition, the rectangle of thoſe perpendiculars ſhall always be equal to a certain given ſpace?

SOLUTION, *by Mr.* LOWRY, *the Propoſer.*

LEMMA. If four points A, B, C, D, (fig. 133, pl. 7,) be taken in a straight line, so that the rectangle AB × AD may be equal to the square of AC, and let AE be taken equal to AC:

Then the line ED is harmonically divided in the points E, B, C, D.

And, conversely, if the line be harmonically divided in these points, and EC be bisected in A, the square of AC is equal to the rectangle $AB \times AD$.

Because $AB \times AD$ is $= AC^2$, $AB : AC = AE :: AC = AE : AD$, and by composition and division,

$$EB : AE :: ED : AD,$$
$$BC : AE :: CD : AD;$$

therefore, $EB : ED :: BC : CD$, and the line ED is harmonically divided in the points E, B, C, D.

Conversely. Because $EB : ED :: BC : CD$, or $EB : BC :: ED : CD$, we have by division and composition,

$$2AB : BC :: 2AE = 2AC : CD,$$
$$2AE = 2AC : BC :: 2AD : CD;$$

therefore, $AB : AC :: AC : AD$, and $AB \times AD = AC^2$.

This being premised. Let EDF, (fig. 134, pl. 7,) be the given ellipse, and AB the straight line given by position; and suppose that the point P is found, such, that if any straight line be drawn through it to meet the curve in E and F, and EG, FH be drawn perpendicular to AB, the rectangle $EG \times FH$ may be equal to a given space, which space is to be found.

In the first place let P be within the ellipse, and draw PQ perpendicular to AB: Then since the proposition is affirmed to be true for every position of EF, it must be true when EF is parallel to AB; and in this case, the rectangle of the perpendiculars is equal to the square of PQ; therefore, $EG \times FH$ must be equal to PQ^2, or PQ^2 must be the given space. Let FE be produced to meet AB in L; then by similar triangles,

$$EG : EL :: PQ : PL,$$
$$\text{and } FH : FL :: PQ : PL;$$
$$\text{therefore } EG \times FH : EL \times FL :: PQ^2 : PL^2:$$

but $EG \times FH$ is $=$ to PQ^2, and therefore $EL \times FL$ is $= PL^2$.

On EL produced, take $LM = PL$; then by the lemma, the line MF is harmonically divided in the points M, E, P, F: wherefore if MN be drawn parallel to AB to meet the diameter DC, drawn through P in N, the position of the point P must be such, that any straight line drawn through it to meet the curve in E and F, and the line NM in M, shall be harmonically divided in the points M, E, P, F.

Now, we know, from the writers on Conic Sections, that this will be the case when NM is parallel to the ordinate of the diameter DC, and when the points P, N are so situated that the line NC is harmonically divided in the points N, D, P, C, that is, by the lemma, (since ON is = OP), when OD × OC is = OP^2.

Secondly, when the point P is without the ellipse.

In this case the given space may be determined by considering the extreme case, when the line drawn through P is a tangent to the curve. Let lines be drawn from P, (fig. 185, pl. 7,) to touch the curve at K and O, and draw KO to meet PC in N, and PF in M. Also, let KV, MQ and OW be drawn perpendicular to AB. Then when the line PEF coincides with PK, the points E and F will coincide with K, and the rectangle of the perpendiculars will be equal to KV^2. In like manner, when PEF coincides with PO, the rectangle of the perpendiculars will be = to OW^2; but since the rectangle of the perpendiculars is = to the same given space in all cases of the proposition, the square of KV must be equal to the square of OW, or KV = OW; therefore KO must be parallel to AB, and the rectangle EG × FH = to MQ^2.

From similar triangles, as in the preceding case, we have EL × FL = LM^2; and because OK is the line which joins the points of contact of the tangents drawn from P, the line PF is harmonically divided in the points P, E, M, F; therefore, by the lemma, PM is bisected in L, and consequently PN is bisected in I: Also, PC is harmonically divided in the points P, D, N, C; therefore, by the lemma, ID × IC is = IN^2 = IP^2. Hence we have this construction for both cases; draw the diameter DC so, that its ordinates may be parallel to AB, and let it meet AB in I. Take IP a mean proportional between ID and IC; then P will be the point required, and PQ^2 the given space.

The demonstration is evident from the analysis.

The same analysis and construction will also apply to the hyperbola; but it is necessary to remark that the straight line given by position, must always meet the hyperbola; whereas in the ellipse, it must always be without, and in the parabola a tangent to the curve. In this case any point whatever in the diameter which passes through the point of contact, will answer the conditions of the proposition.

VIII. QUESTION 218, *by Mr.* LOWRY.

If the vertical angle and ſum of the ſides of a plane triangle remain conſtant, what is the equation of the curve to which the baſe is always a tangent?

SOLUTION

Solution, *by Mr.* Lowry, *the Proposer.*

Let the straight lines AG, AH, (fig. 136, pl. 7,) include the given angle, and let GDH be the curve required, such, that if a tangent be drawn at any point D, meeting the lines AG, AH in B and C; the sum of AB, AC may be equal to a given line a.

Draw DE, DF parallel to AH, AG; and put x = AE = DF, and y = DE = AF; then BE is $= -\frac{y\dot{x}}{\dot{y}}$ and FC $= -\frac{x\dot{y}}{\dot{x}}$, and we have $AB + AC = x - \frac{y\dot{x}}{\dot{y}} + y - \frac{x\dot{y}}{\dot{x}} = a$,

$$\text{or } x - \frac{y\dot{x}}{\dot{y}} + y - \frac{x\dot{y}}{\dot{x}} - a = 0,$$

the fluxionary equation of the curve.

This equation is of that singular kind that admits of two fluents or integrals, namely, a complete or general integral, containing a constant arbitrary quantity; and a particular integral into which no arbitrary quantity enters, and which is not comprised in the general integral*. This class of equations is generally resolved by first taking the fluxion, considering the fluxion of one of the variables as constant; thus, in the present case, by taking the fluxion of the above equation, considering $\dot{x}$ as constant, we have

$$\dot{x} - \frac{\dot{y}^2\dot{x} - y\dot{x}\ddot{y}}{\dot{y}^2} + \dot{y} - \frac{\dot{x}\dot{y} + \ddot{y}x}{\dot{x}} = 0,$$

$$\text{and by reduction, } \left(\frac{\dot{x}y}{\dot{y}^2} + \frac{x}{\dot{x}}\right)\ddot{y} = 0;$$

therefore, either $\frac{\dot{x}y}{\dot{y}^2} + \frac{x}{\dot{x}}$ must be $= 0$, or $\ddot{y} = 0$.

If $\frac{\dot{x}y}{\dot{y}^2} + \frac{x}{\dot{x}}$ be $= 0$, then $\frac{\dot{x}}{\dot{y}} = -\sqrt{(\frac{x}{y})}$; and this value of $\frac{\dot{x}}{\dot{y}}$ being substituted in the original fluxionary equation it becomes $x + y + y\sqrt{\frac{x}{y}} + x\sqrt{\frac{y}{x}} - a = 0$, from whence we deduce $\sqrt{x} + \sqrt{y} = \sqrt{a}$, or $(a - x) + y = 2\sqrt{ay}$, an equation to the common parabola, and which is the curve required. As this equation does not contain any constant arbitrary quantity, it is necessarily the particular integral equation. We shall now shew how, from the equation $\ddot{y} = 0$, the complete integral may be derived; and here, because $\dot{x}$ is constant, we have by integration, $\dot{y} = A\dot{x}$, and $y = B + Ax$, where B is a constant arbitrary quantity, (namely, the value of y when $x = 0$,) and A a quantity whose value is dependant on that of B. Now when x is $= 0$, y is $= \frac{a}{1 - \frac{\dot{x}}{\dot{y}}} = \frac{a}{1 - \frac{1}{A}} =$ (by hypothesis) to B; or if B be put $= \frac{a}{b}$ (to render the expressions homogeneous), we have $\frac{a}{1 - \frac{1}{A}} = \frac{a}{b}$, or $A = -\frac{1}{b - 1}$ and the complete integral equation becomes $y = \frac{a}{b} - \frac{1}{b - 1}x$, or $\frac{b^2}{2}y - by - b(a - x) + 2a = 0$, an equation to a straight line.

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* On the subject of particular integrals, see an ingenious work by Mr. Woodhouse, of Cambridge, entitled "The Principles of Analytical Calculation," page 97. And "Leçons sur le Calcul des Fonctions," par M. La Grange.

Indeed it is evident, that the straight line BC, will satisfy the conditions of the problem, provided it be placed so that AB + AC may be equal to a: that is, if b be taken of any value from 1 to ∞, and AC be made $= \frac{a}{b}$, and AB $= a - \frac{a}{b}$; then AB + AC will be equal to a, and the line BC will be the locus of the fluxionary equation $x - y\frac{\dot{x}}{\dot{y}} + y - x\frac{\dot{y}}{\dot{x}} - a = 0$, as well as the parabola GDH. In fact the parabola is formed by the continual intersections of the straight lines represented by the complete integral equation, when the arbitrary quantity b is made continually to vary.

Though the particular integral cannot be derived from the general one, by assigning any particular value to the arbitrary quantity b; yet it may be deduced from it by considering the quantity b as variable, or a function of x and y.

Thus, if in the complete integral equation $\frac{b^2}{2}\,y - by - b(a - x) + 2a = 0$, we take the fluxion, supposing b only to vary, we have $b = 1 + \frac{a - x}{y}$; this value of b being substituted in the given equation, it becomes, after proper reduction,

$$(a - x)^2 + 2y\,(a - x) + y^2 = 4ay,$$

or $(a - x) + y = 2\sqrt{ay}$, the particular integral as determined before.

This question was also answered by Dr. O'Riordan.

IX. QUESTION 219, *by Mr.* WALLACE.

Let AB, AC be two ſtraight lines given by poſition, and P a given point: it is required to draw a line through P to meet the given lines in D and E, ſo that AD × AE may be to DP × PE in a given ratio.

FIRST SOLUTION, *by Dr.* O'RIORDAN.

Suppose that DE, (fig. 137, pl. 7,) is drawn as required, and let PI, PH be drawn perpendicular to AB, BC: and AQ perpendicular to DE; then PI, PH are given lines, and since

AD × IP = DP × AQ = twice the triangle DPA,
and AE × PH = PE × AQ = twice the triangle PAE,
we have AD : DP :: AQ : IP, and AE : PE :: AQ : PH; therefore AD × AE : DP × PE :: AQ^2 : IP × PH, or AQ^2 has to IP × PH the given ratio of AD × AE to DP × PE: Let this ratio be taken = that of PS to PI, and then AQ^2 is = to the given space PS × PH. Hence this construction: On AP describe a semicircle, and therein apply the line AQ = a mean proportional between PS and PH: and through the points P and Q draw DE.

SECOND SOLUTION, *by Mr.* CUNLIFFE.

Join PA, (fig. 138, pl. 7,) and draw PQ parallel to AB meeting CA in Q. Put AQ = a, QP = b, AE = x, and

$= y$: also let the given ratio of AD × AE to PD × PE be denoted by that of b to c. Then because of the parallels QP, AB,

$$QE : QP :: AE : AD = QP \times AE \div QE;$$

$$\text{whence } AD \times AE = QP \times AE^2 \div QE:$$

$$\text{Also } QE : PE :: QA : PD = QA \times PE \div QE;$$

whence $PD \times PE = QA \times PE^2 \div QE$; and therefore $AD \times AE : PD \times PE :: QP \times AE^2 : QA \times PE^2 :: b : c$: Whence $c \times QP \times AE^2 = b \times QA \times PE^2$, that is, $bcx^2 = aby^2$, or $cx^2 = ay^2$.

Hence it appears, that x to y, or AE to PE have a given ratio, which makes the construction very easy, and may be as follows:

Take two right lines Ae and ep in the given ratio of AE to EP, and set off Ae upon AC; from e to AP apply ep, and through P draw PDE parallel to pe cutting AB, AC in D and E, and the thing is done.

X. QUESTION 220, *by Mr.* LOWRY.

If through a given point P on the ſurface of a ſphere, two great circles be drawn to interſect a ſmall circle given by poſition in the points A, B and C, D; and the points AC, BD be joined by great circles which interſect in Q. What is the locus of the point Q?

SOLUTION, *by Mr.* LOWRY, *the Proposer.*

Let O, (fig. 139, pl. 7,) be the centre of the sphere, and ABCD the small circle given by position. Conceive lines to be drawn from the points A, B, C, D to the centre of the sphere, and let the planes of the great circles PAB, PDC, AQC, BQD meet the plane of the small circle in the straight lines GAB, GCD, AIC and DIB respectively. Then because P is a given point, G is also a given point; and, since straight lines are drawn from G to meet the circle in the points A, B and A, D, the locus of the intersection I of the lines AC, BD is a straight line given by position; namely, the line EF which joins the points of contact of tangents drawn to the circle from G. This property of the circle is too generally known to need demonstrating here. Let a plane pass through the line EF and the centre O; then since the point I is always in the line EF, the planes AIC, BID will always intersect in the plane EFO; therefore the point Q must be in the curve formed by the intersection of the plane OEF with the superficies of the sphere; that is, in the great circle which passes through the points E, F. And since the planes GEO, GFO

touch

touch the circle at E, and F, the arches PE, PF, which are in these planes, will also touch the small circle in these points; therefore if great circles be drawn from the given point to touch the small circle in the points E and F, a great circle described through these points is the locus required.

It is obvious that the great circles joining the points A, D; B, C will also intersect in the great circle EQF.

In the figure referred to, the given point is without the circle, but the same reasoning will apply when it is within the circle; only the locus will then be on the other side of S. Thus, if Q be the given point, the locus of P will be a great circle drawn perpendicular to the diameter which passes through Q.

Conceive a plane passing through the points G, O, and any point I, in the line EF, to meet the plane of the circle in the straight line GSIT, and the surface of the sphere in the great circle PSQT; then it is well known, that the line GT is harmonically divided in the points G, S, I, T, or GS : GT :: SI : IT; we shall now shew that the arch PT is divided, so that sin PS : sin PT :: sin SQ : sin QT.

This division may, for the sake of abridging language, be called the harmonical division of the arch PT.

Let lines be drawn from the points S, Q, T to the centre of the sphere: Then by trigonometry,

$$GS : SO :: \sin GOS : \sin OGS,$$

$$OT = SO : GT :: \sin OGS : \sin TOG; \text{ therefore ex æquo,}$$

$$GS : GT :: \sin GOS : \sin TOG :: \sin PS : \sin PT.$$

In the same way, it is shewn, that

$$SI : IT :: \sin SQ : \sin QT.$$

Therefore sin PS : sin PT :: sin SQ : sin QT.

This harmonical division of the arch PT suggests a number of curious properties analogous to those which are connected with the harmonical division of the diameter of a circle in *plano*. The following are some of the most remarkable.

On the great circle AB, (fig. 140, 141, pl. 7,) which passes through the pole C of a small circle on the surface of the sphere, let two points D, P, be taken, so, that sin PA : sin PB :: sin AD : sin DB, or which is the same thing, (as is easily proved,) that $\tan PC \times \tan CD$ may be $= \tan^2 CA$, and through D let a great circle EF be drawn perpendicular to AB: Then,

1st. If any great circle be drawn through P to meet the small circle in S and T, and the great circle EF in Q, the arc PT is harmonically divided in the points P, S, Q, T.

2d.

2nd. If great circles be drawn through the points S, D; T, D to meet the small circle in G and K; the angle PDS is = BDT; and the angle SPD = DPG.

3rd. The points P, G, K are in the arch of a great circle.

4th. The arch AS is = to the arch AG, and the arch ST = to the arch GK.

5th. Sin PS : sin PT :: sin SD : sin DT.
and sin PS : sin PK :: sin SD : sin DK.

6th. If great circles be drawn to touch the small circle at S and T, they will intersect in the great circle EF.

Dr. O'Riordan *also answered this question.*

XI. QUESTION 221, *by Mr.* LOWRY.

Find the perſpective repreſentation of a given ſpheroid placed on a horizontal plane; the relative poſitions of the eye, the vertical plane of the picture, and the ſpheroid being given?

SOLUTION, *by Mr.* LOWRY, *the Propoſer.*

Let E, (fig. 142, pl. 7,) be the position of the eye, and C the centre of the given spheroid. Conceive a straight line to be drawn through the points E, C to meet the surface of the spheroid in A and B; and let a point O be found in AC, such, that the rectangle $OC \times CE$ may be equal to AC^2. Through O, let a a plane be drawn parallel to the tangent plane at A, meeting the surface of the spheroid in the curve IHGF, which, it is well known, is an ellipse, except in the particular case when the plane is perpendicular to AB, and then it is a circle. Let AGB be any section of the spheroid, made by a plane passing through the eye, and the centre, and meeting the ellipse (or circle) IGHF in G; and join EG, GO. Then because the plane HGOF is parallel to the tangent plane at A, it is evident that GO will be parallel to the tangent of the ellipse AGB at A; and since the rectangle $OC \times CE$ is = to AC^2, the line EG will touch the ellipse AGB at G, (Hamilton's Conic Sections, b. 1, p. 48). Whence it is evident that the ellipse IHGF is the locus of the points of contact of all the tangents drawn to the spheroid from E; or it is the curve of contact formed on the surface of the spheroid by the visual rays from E; and consequently its representation on the given plane will form the contour of the perspective of the spheroid. Now it is shewn by the writers on perspective, that the representation of any ellipse, on a given plane, is a conic section, and therefore the perspective representation of the given spheroid will also be a conic section; namely, an ellipse when the spheroid is without the directing plane; a parabola when it touches the directing plane on the side next to the picture; and a hyperbola when it is cut by the directing plane.

Though it may be difficult to describe the ellipse HGFI, on the surface of the spheroid, so that the common rules for putting an ellipse into perspective, may be applicable; yet we may easily determine as many points in the curve as shall be necessary for describing its representation on the given plane.

Let X represent the greater, and Y the less axis of the spheroid when drawn through the centre, parallel to the horizon; and conceive a vertical plane passing through E and C, to cut the spheroid in the elliptic section AFBH: Then because the point E, and the spheroid, are given by position, the vertical plane AFBH will make a given angle with the vertical plane, passing through the axis X, and therefore the ellipse AFBH will be given in magnitude and position. And, if tangents EF, EH be drawn from E, we may easily determine the points of contact F, H by first describing the ellipse on a plane, and drawing the tangents, and then transferring the points of contact to the surface of the spheroid. In like manner, if a plane pass through E, and each of the axes X, Y, cutting the spheroid in two elliptic sections, the points of contact of tangents drawn to these sections from E, may be found as before, and we shall then have six given points in the ellipse IHGF to describe its perspective; and the representation of any five of them being made on the plane of the picture, we have only to describe a conic section through these five points and it will be the figure required.

A solution was also received from Dr. O'Riordan.

XII. QUESTION 222, *by Mr.* BAZLEY.

It is required to determine the locus of the vertex of a plane triangle, whereof the radius of the inſcribed circle, and difference of the angles at the baſe are given; the middle of the baſe being a fixed point in a ſtraight line given by poſition?

FIRST SOLUTION, *by Mr.* BAZLEY, *the Proposer.*

CONSTRUCTION. Draw the indefinite right line MN, (fig. 143, pl. 7,) wherein take E for the middle of the base. Erect the perpendicular EF = to the diameter of the inscribed circle and make the ∠ FEQ = ½ the difference of the angles at the base:

through F draw FVS parallel to MN cutting EQ in V; take VG (in VQ) $= \frac{1}{2}$EV, and GH, parallel to MN, $= \frac{1}{2}$FV. Lastly, to the asymptotes VQ, VS, through the point H, describe the hyperbola HR, which is the locus required.

DEMONSTRATION. Take any point C in HR, and draw CTOI parallel to EVQ, cutting VS in T, and MN in I; bisect TI in O, and drop the perpendicular OL on MN: with radius OL and centre O, describe the circle OL, and from C draw CA, CB touching it, and meeting MN in A and B. Then it is plain, that the triangles IOL, VEF are similar, and since IO $= \frac{1}{2}$IT $= \frac{1}{2}$EV, therefore OL $= \frac{1}{2}$EF: whence it is evident, that in the triangle ABC, OL is the given inscribed circle; and it is well known that IOL (=FEV) $= \frac{1}{2}$ the difference of the angles at the base. Now it only remains to prove that E is the middle of AB.

By the property of the hyperbola VT × TC = VG × GH, that is, by construction, EI × TC = IO × IL, or EI : IL :: IO : TC, and by composition, EI : EL :: (IO : OC) IL : LK; and, again, by composition, EI : EL :: EL : EK; therefore by a known property, (whose demonstration may be seen at pa. 285, vol. 1, Old Series of the Repository,) E is the middle of the base AB.

SECOND SOLUTION, *by Dr.* O'RIORDAN.

Let ACB, (fig. 144, pl. 7,) be a triangle circumscribed by a circle, EF being the diameter which bisects the base AB in M: Draw EC, EB: upon EC take E*s* = EB, and draw *st* perpendicular to AB. Then the following things are pretty generally known; namely, *s* is the centre, and *st* the radius of the circle inscribed in the triangle ACB; also $\frac{1}{2}(\angle \text{ABC} - \angle \text{BAC}) = \angle \text{FEC} = \angle \text{L}st$ which is given by the question. Therefore put $\text{L}t = a$, $\text{L}s = b$, $\text{ML} = x$, and $\text{LC} = v$. Then per similar triangles L*ts*, LME,

$$\text{L}t : \text{L}s :: \text{ML} : \text{EL} = (\text{L}s \times \text{ML}) \div \text{L}t = bx \div a;$$

whence $\text{E}s = \text{EB} = \text{EL} + \text{L}s = \frac{bx}{a} + b$, $\text{EC} = \text{EL} + \text{LC} = \frac{bx}{a} + v$.

Again, per similar triangles, ELB, EBC, EL : EB :: EB : EC;

whence $\text{EL} \times \text{EC} = \text{EB}^2 = \text{E}s^2$, that is $\frac{bx}{a} \times \frac{bx + av}{a} = \frac{b^2}{a^2} \times (x + a)^2$

or $x\ (bx + av) = b\ (x + a)^2$, which being reduced, gives $vx - 2bx = ab$.

Put $y = v - 2b$, and the equation becomes $xy = ab$, which expresses the known property of the co-ordinates of a hyperbola, with respect to its asymptotes; therefore the locus of the vertex C of the triangle is an hyperbola: The position of the asymptotes, &c. may be determined as follows:

Draw MP parallel to LC, on LC take $sl = sL$, and through l draw RQ parallel to AB, meeting MP in R. Then $RL = ML = x$, and $lC = LC - lL = v - 2b = y$; therefore $xy = ab$, or $Rl \times lC = Lt \times Ls$, which is the known property of an hyperbola, whose asymptotes are RP, RQ, and centre R. Suppose $x = y$, then $xy = y^2 = ab$, or $x = y = \sqrt{ab}$, which determines the place of the principal vertex.

XIII. QUESTION 223, *by* ———.

Let a circle be given by poſition, and let A and B be two given points, a point C may be found, ſuch, that if any ſtraight line be drawn through C to meet the circle in D and E, and AD, BD, AE, BE be joined, $AD^2 + DB^2 : AE^2 + EB^2 :: DC : CE$. Required the demonſtration?

FIRST SOLUTION, *by Mr.* I. POUND.

ANALYSIS. Join AB, (fig. 145, pl. 7,) which bisect in G, and join DG, EG; then $AD^2 + DB^2 = 2AG^2 + 2DG^2$ and $AE^2 + EB^2 = 2GE^2 + 2GB^2$; and if GC′, (C′ being the centre of the circle,) be joined, meeting the circle in H and K, and $GL^2 = HLK$, and DM, EN be perpendicular to GC′: Then by the 4th Porism of Euclid's 3rd. Book of Porisms, $DG^2 = 2GC' \times LM$, and $GE^2 = 2GC' \times LN$. Now make $4AG^2 = 2GC' \times LO$, then $2AG^2 + 2DG^2 = 4GC' \times OM$ and $2GE^2 + 2GB^2 = 4GC' \times ON$; therefore $OM : ON :: DC : CE$, that is, drawing RO perpendicular, and DQ, EP, parallel to CG′, $DQ : PE :: DC : CE$; but if DE meet OR in R, $DQ : PE :: DR : RE$; therefore, finally, $DR : RE :: DC : CE$.

Whence it is evident, that C is such a point in OC′ that $HO : OK :: HC : CK$. *Q. E. I.*

The construction and demonstration are evident from the analysis.

N. B.—It is evident that the porism is true for any figure given in species as well as squares.

SECOND SOLUTION, *by Mr.* LOWRY.

By making the same construction and substitution, as in the solution to quest. 216, (fig. 132, pl. 7,) we have $AD^2 + BD^2 = 4a(h + x)$, and $AE^2 + BE^2 = 4a(h + x')$; therefore, by the question,

$$4a\ (h + x) : 4a\ (h + x') :: CD : CE,$$
$$\text{or } h + x : h + x' :: CD : CE;$$

and by composition and division,

$$(h + x) + (h + x') : (h + x) - (h + x') :: CD + CE : CD - CE.$$

But it is evident, from the solution alluded to, that

$$(h + x) + (h + x') = 2h + a - 2y \sin^2 \varphi,$$
$$(h + x) - (h + x') = 2\cos\varphi \sqrt{(r^2 - y^2 \sin^2 \varphi)}$$
$$CD + CE = 2\sqrt{(r^2 - y^2 \sin^2 \varphi)}$$
$$\text{and} \qquad CD - CE = -\frac{2b}{\cos\varphi} + 2y \cos\varphi;$$

therefore,

$$(h + \frac{a}{2} - y \sin^2 \varphi)\ (y \cos\varphi - \frac{b}{\cos\varphi}) = \cos\varphi\ (r^2 - y^2 \sin^2 \varphi),$$

$$\text{or } (h + \frac{a}{2})\ y \cos^2 \varphi - (h + \frac{a}{2} - y \sin^2 \varphi]\ b = r^2 \cos^2 \varphi;$$

in which equation, the angle φ must be indeterminate; which it evidently will be when b is $= 0$; for then y is $= c$, and the equation becomes $(h + \frac{1}{2}a)\ c \cos^2 \varphi = r^2 \cos^2 \varphi$, or $(h + \frac{1}{2}a)\ c = r^2$. Therefore the point C is in the line GF, and at a distance from G $=$ to $\dfrac{r^2}{h + \frac{1}{2}a}$.

THIRD SOLUTION, *by Mr.* CUNLIFFE.

Draw the line AB, (fig. 146, pl. 7,) which bisect in M, and draw MD, ME; also, draw MO, to the centre O of the given circle. By a known property

$AD^2 + BD^2 = 2(AM^2 + MD^2)$ and $AE^2 + BE^2 = 2(AM^2 + ME^2)$;

therefore,

$AD^2 + BD^2 : AE^2 + BE^2 :: AM^2 + MD^2 : AM^2 + ME^2 :: DC : CE.$

Now a little attention to the more obvious circumstances, of the question will be sufficient to shew, that if there is such a point C as that asserted, it must be situated somewhere in the line MO; for when DC = CE, then it is manifest from the preceding analogy, that MD = ME, and therefore the angles OCD, MCD are right-angles Euc. 3. III, and consequently the points O, C, M, are in the same right line.

This consideration will tend very much to simplify the investigation. Draw the radii OD, OE; also draw Dn, Em perpendicular to MO.

Put OM $= a$, OC $= b$, AM = BM $= c$, OD = OC $= r$, ON $= x$, and OM $= y$. Then, Mn $= a - x$, Mm $= a + y$, $Dn^2 = r^2 - x^2$, $Em^2 = r^2 - y^2$, $MD^2 = Mn^2 + Dn^2 = (a-x)^2 + r^2 - x^2 = a^2 + r^2 - 2ax$,
$ME^2 = Mm^2 + Em^2 = (a+y)^2 + r^2 - y^2 = a^2 + r^2 + 2ay$: hence
$AM^2 + MD^2 = a^2 + r^2 + c^2 - 2ax = s^2 - 2ax$, and
$AM^2 + ME^2 = a^2 + r^2 + c^2 + 2ay = s^2 + 2ay$, putting $s^2 = a^2 + r^2 + c^2$.
Now, per similar triangles, CnD, CmE, DC : CE :: Cn : Cm; therefore, $AM^2 + MD^2 : AM^2 + ME^2 ::$ Cn : Cm, whence
$Cm \times (AM^2 + MD^2) = Cn \times (AM^2 + ME^2)$, that is
$(b + y) \times (s^2 - 2ax) = (x - b) \times (s^2 + 2ay)$, which gives

$$x = \frac{2bs^2 + (s^2 + 2ab)\,y}{s^2 + 2ab + 4ay}.$$

Again, per similar triangles, CnD, CmE, $Dn^2 : Em^2 :: Cn^2 : Cm^2$; whence
$Dn^2 \times Cm^2 = Em^2 \times Cn^2$, or $(r^2 - x^2) \times (b + y)^2 = (r^2 - y^2) \times (x - b)^2$

which gives $x = \dfrac{2br^2 + (r^2 + b^2)\,y}{r^2 + b^2 + 2by}$.

Therefore, $\dfrac{2bs^2 + (s^2 + 2ab)\,y}{s^2 + 2ab + 4ay} = \dfrac{2br^2 + (r^2 + b^2)\,y}{r^2 + b^2 + 2by}$,
which cleared of fractions, gives $2bs^2(r^2 + b^2) + (r^2 + b^2) \times (s^2 + 2ay)\,y + 4b^2s^2y + 2by^2(s^2 + 2ab) = 2br^2(s^2 + 2ab) + (r^2 + b^2)(s^2 + 2ab)\,y + 8abr^2y + 4ay^2(r^2 + b^2)$,
which equation will, in general, give a determinate value of y, but we want such a relation of the other quantities as will make y indeterminate; and this will be done by equating the like terms on each side. Beginning with the co-efficients of y^2, $b\,(s^2 + 2ab) = 2a\,(r^2 + b^2)$; whence $b = 2ar^2 \div s^2$: And the coefficients of y being equated, give exactly the same result; as do also the remaining terms.

XIV. QUESTION 224, *by Mr.* IVORY.

Suppoſing x to denote any fraction leſs than unit, it is required to aſſign the value of the infinite product $(1 + x)$ $(1 + x^2)$ $(1 + x^4)$ $(1 + x^8)$ &c.?

FIRST

First Solution, *by Mr.* J. Wallace, *teacher of Mathematics, Edinburgh.*

Let P be the product required. Then, by the logarithmic series, we have

$$\begin{aligned}
&\text{Log. }(1+x\ ) = \text{M}(x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\tfrac{1}{5}x^5-\tfrac{1}{6}x^6+\tfrac{1}{7}x^7-\tfrac{1}{8}x^8\ \&\text{c.}\\
&\text{Log. }(1+x^2) = \text{M}(\quad +x^2 \qquad -\tfrac{1}{2}x^4 \qquad +\tfrac{1}{3}x^6 \qquad -\tfrac{1}{4}x^8\ \&\text{c.}\\
&\text{Log. }(1+x^4) = \text{M}(\qquad\qquad +x^4 \qquad\qquad -\tfrac{1}{2}x^8\ \&\text{c.}\\
&\text{Log. }(1+x^8) = \text{M}(\qquad\qquad\qquad\qquad\qquad +x^8\ \&\text{c.}\\
&\qquad \&\text{c.} \qquad\qquad \&\text{c.} \qquad\qquad \&\text{c.}
\end{aligned}$$

Hence, taking the sum, we have

$$\text{Log. P} = \text{M}\ (x+\tfrac{1}{2}x^2+\tfrac{1}{3}x^3+\tfrac{1}{4}x^4+\tfrac{1}{5}x^5+\tfrac{1}{6}x^6+\tfrac{1}{7}x^7+\tfrac{1}{8}x^8+\&\text{c.})$$

But M $(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \&\text{c.})$ is known to be the logarithm of $\frac{1}{1-x}$. Hence it follows that $\text{P} = \frac{1}{1-x}$.

Second Solution, *by Mr.* Wallace, *R. M. College.*

$$\begin{aligned}
\text{Because}\qquad (1+x)\ (1-x) &= 1-x^2\\
(1+x^2)\ (1-x^2) &= 1-x^4\\
(1+x^4)\ (1-x^4) &= 1-x^8\\
\ldots\ldots\ldots\ldots\ldots\ldots &\ldots\ldots\ldots\\
(1+x^n)\ (1-x^n) &= 1-x^{2n},
\end{aligned}$$

therefore taking the products of the two sides of these identical equations, and leaving out the factors common to both, we get

$$(1+x)\ (1+x^2)\ (1+x^4)\ldots\ldots\times(1+x^n)\ (1-x) = 1-x^{2n},$$

and hence,

$$(1+x)\ (1+x^2)\ (1+x^4)\ldots\ldots\times(1+x^n) = \frac{1-x^{2n}}{1-x}.$$

This is true, whatever be the number of factors in the product. If, however, we suppose that number to be infinite, so that n is infinitely great, then $x^{2n} = 0$, (x being supposed less than unity), and consequently,

$$(1+x)\ (1+x^2)\ (1+x^4)\ (1+x^8)\ \&\text{c.} = \frac{1}{1-x}.$$

Third Solution, *by* ———.

Let $f\ (x) = (1+x)\ (1+x^2)\ (1+x_4)\ \&\text{c.}$

Then $f\,(x^2) = (1+x^2)\ (1+x^4)\ (1+x^8)\ \&\text{c.}$

Therefore $f(x) = (1+x)\, f(x^2)$,

and $(1-x)\, f(x) = (1-x^2)\, f(x^2)$.

In like manner,

$$(1-x^2)\, f(x^2) = (1-x^4)\, f(x^4)$$
$$(1-x^4)\, f(x^4) = (1-x^8)\, f(x^8)$$
$$(1-x^8)\, f(x^8) = (1-x^{16})\, f(x^{16})$$

&c. &c.

Therefore, ultimately, $(1-x)\, f(x) = 1$,

and $f(x) = \frac{1}{1-x}$.

Fourth Solution, *communicated by* Philalethes Cantabrigiensis.

About twenty years have elapsed since the person alluded to in page 56, of vol. 1, of the *New Series* of this *Repository*, produced the following equations, viz.

Let x be less than 1, then by multiplying both numerator and denominator by equal quantities, we shall have

$$\frac{1}{1-x} = \frac{1+x}{1-xx} = \frac{(1+x)(1+xx)}{1-x^4} = \frac{(1+x)(1+xx)(1+x^4)}{1-x^8}, \text{ \&c.}$$

Hence, putting n to denote any power of 2, which exceeds 8, we have

$$\frac{1}{1-x} = \frac{(1+x)(1+xx)(1+x^4)(1+x^8)\ldots\ldots\ldots\ldots(1+\overline{xx}|^{n})}{1-\overline{xx}|^{2n}};$$

and this equation, when n is infinite, becomes

$$\frac{1}{1-x} = (1+x)(1+xx)(1+x^4)(1+x^8)(1+x^{16}), \text{ \&c. } \textit{ad infi-}$$

nitum, the term $\overline{xx}|^{2n}$, in that case, becoming less than any assignable quantity, and consequently the denominator may be considered as $= 1$.

Corol. 1. It is evident that any power, or root, of the fraction on the first side of the equation now obtained, is equal to the continued product of the same power, or root, of all the factors on the second side of it.

Corol. 2. It is evident also, that the

Logarithm of $\frac{1}{1-x} = \text{L.}\,(1+x) + \text{L.}\,(1+xx) + \text{L.}\,(1+x^4)$, &c. *ad infinitum.*

Scholium. Several other useful deductions may be made from the above equation; which equation, I observe, may easily be deduced from what was shown by the Rev. *John Hellins*, in the *Philoſophical Tranſactions* of the Royal Society of *London*, for the year 1798, page 185.

XV QUESTION 225, *by Mr.* CUNLIFFE.

To find integer values for the ſides of a trapezium that may be inſcribed in a circle, ſuch, that its diagonals, diameter of the circumſcribing circle, and area, may all be expreſſed by whole numbers?

SOLUTION, *by Mr.* CUNLIFFE, *the Propoſer.*

Let ABCE, (fig. 147, pl. 7,) be a trapezium inscribed in a circle, AC and BE the diagonals intersecting in M, and BF perpendicular to AC. Put AB $= m^2 + n^2$, AF $= m^2 - n^2$; BM $= m'^2 + n'^2$, MF $= m'^2 - n'^2$; BC $= m''^2 + n''^2$, CF $= m''^2 - n'^2$; whence BF $= 2mn = 2m'n' = 2m''n''$.

Take BF $= 24$, then $mn = m'n' = m''n'' = 12 \times 1 = 4 \times 3 = 6 \times 2$; therefore, we may take $m = 12, n = 1$; $m' = 4, n' = 3$; $m'' = 6, n'' = 2$: then AB $= m^2 + n^2 = 145$, AF $= m^2 - n^2 = 143$, BM $= m'^2 + n'^2 = 25$, MF $= m'^2 - n'^2 = 7$; BC $= m''^2 + n''^2 = 40$, CF $= m''^2 - n''^2 = 32$; and hence AC $= 175$, AM $= 136$ and CM $= 39$.

And by the property of the circle, EM $= \frac{\text{AM} \times \text{MC}}{\text{BM}} = \frac{136 \times 39}{25}$, hence BE $=$ BM $+$ ME $= \frac{5929}{25}$: and per similar triangles, BMA, CME; BM : BA :: CM : CE $= \frac{\text{BA} \times \text{CM}}{\text{BM}} = \frac{145 \times 39}{25} = \frac{1131}{5}$. Also, per similar triangles, CMB, EMA; BM : BC :: AM : AE $= \frac{\text{BC} \times \text{AM}}{\text{BM}} = \frac{40 \times 136}{25}$ $=$

$= \frac{1088}{5}$. Again, by a known property, $\frac{AB \times BC}{BF} = \frac{145 \times 40}{24}$ $= \frac{725}{3} =$ the diameter of the circumscribing circle.

And by reducing the preceding numbers to a common denominator, and omitting it, we shall have $BF = 24 \times 75 = 1800$, $AB = 145 \times 75 = 10875$, $BC = 40 \times 75 = 3000$, $AC = 175 \times 75 = 13125$, $BE = 17787$, $CE = 1131 \times 15 = 16965$, $AE = 1088 \times 15 = 16320$, and the diameter of the circumscribing circle $= 725 \times 25 = 18125$.

The area of the trapezium may be obtained as follows: $AC \times \frac{1}{2}BF = 13125 \times 12 = 157500 =$ area of the Δ ABC. And it is pretty evident that $BM : BE :: \Delta ABC :$ trapez. $ABCE = \frac{BE \times ABC}{BM} = \frac{17787 \times 13125 \times 12}{25 \times 75} = 17787 \times 7 \times 12 = 1494108$.

XVI. QUESTION 226, *by Mr.* CUNLIFFE.

Suppoſe a heavy hollow cylinder, of an uniform thickneſs, to roll freely along an inclined plane of a given length and inclination: having given the external diameter of the cylinder together with the time of rolling along the inclined plane, it is required from thence to determine the internal diameter of the cylinder?

SOLUTION, *by Mr.* CUNLIFFE, *the Proposer.*

Let SAB, (fig. 148, pl. 7,) be a section of the cylinder perpendicular to its axis; C its centre; CS and CR the external and internal radii of the section. Before we proceed to the solution of the question, it will be necessary to determine the centre of oscillation of the section, the point of suspension being in the circumference of the outer circle, and the section supposed to vibrate in a vertical plane edgeways.

Let Cq be an intermediate circle between CS and CR, and suppose O, in the diameter SB, the centre of oscillation of the section with respect to the centre of suspension S.

Put $CS = R$, $CR = r$, $Cq = x$, $p = 3{\cdot}1416$, and let s denote the area of the annulus included between the circles, whose radii are Cq and CR. Then, by Emerson's Fluxions, page 311, $CO = \frac{\int x^2 \dot{s}}{Rs}$ when $x = R$. Now, $x^2 \dot{s} = x^2 \times 2px\dot{x} =$

$2px^3\dot{x}$, the correct fluent of which is $\frac{1}{2}p \times (x^4 - r^4)$; also $\dot{s} = p\,(x^2 - r^2)$; therefore $\frac{\int x^2\dot{s}}{Rs} = \frac{\frac{1}{2}p\,(x^4-r^4)}{Rp\,(x^2-r^2)} = \frac{x^2 + r^2}{2R}$, which, when $x = R$ becomes $\frac{R^2 + r^2}{2R}$, that is, $CO = \frac{R^2+r^2}{2R}$, and $SO = SC + CO = R + \frac{R^2+r^2}{2R} = \frac{3R^2 + r^2}{2R}$.

Solution of the Question.

Let l denote the length, and h the height of the inclined plane, and put $g = 16\frac{1}{12}$; also let t denote the given time of the cylinder's rolling along the inclined plane.

Then, it is well known that $hg \div l$, will express the distance which a heavy body would descend freely from rest upon the inclined plane, in the first second of time. From whence, and by cor. 2, prop. 61, sect. vi, Emerson's Mechanics, $(SC \div SO) \times (hg \div l)$ will express the distance which the cylinder will roll upon the inclined plane in the first second.

Then by the law of uniformly accelerated motion, $\frac{SC}{SO} \times \frac{hg}{l} : l :: 1 : t^2$; whence $\frac{SC}{SO} = \frac{2R^2}{3R^2 + r^2} = \frac{l^2}{hgt^2}$ which gives $r = \frac{R\sqrt{(2hgt^2 - 3l^2)}}{l}$. *Q. E. I.*

This question was also answered by S. C. I.

XVII. QUESTION 227, *by Mr.* CUNLIFFE.

Required the ſum of the infinite ſeries

$$\frac{1}{1 \cdot 3^2} + \frac{2}{3 \cdot 5} + \frac{3}{5 \cdot 7^2} + \frac{4}{7 \cdot 9^2} + \frac{5}{9 \cdot 11^2} + \&c.$$

SOLUTION, *by Mr.* J. WALLACE, *Edinburgh.*

The general term of this series is evidently $\frac{x}{(2x-1)(2x+1)^2}$

And, by applying the common rules for the decomposition of fractions, we find

$$\frac{x}{(2x+1)(2x+1)^2} = \frac{1}{8} \cdot \frac{1}{2x-1} + \frac{1}{4} \cdot \frac{1}{(2x+1)^2} - \frac{1}{8} \cdot \frac{1}{2x+1}.$$

Hence, putting S for the sum of the given series, we have

$$S = \begin{cases} \frac{1}{8}(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \&c. \\ +\frac{1}{8}(-\frac{1}{3} - \frac{1}{5} - \frac{1}{7} - \frac{1}{9} - \&c. \\ +\frac{1}{4}(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c. \end{cases}$$

that is, $S = \frac{1}{8} + \frac{1}{4}(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c.$

But it is known that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c.$$

(Euleri Introduct. in Anal. Infin. § 175). Hence we have

$$S = \frac{1}{8} + \frac{1}{4}(\frac{\pi^2}{8} - 1) = \frac{(3{\cdot}1416)^2}{32} - \frac{1}{8} = {\cdot}1734246.$$

XVIII. QUESTION 228, *by Mr.* WALLACE.

Find the nature of a curve, ſuch, that any normal ſhall have a given ratio to the ſegment of the axis intercepted between it and a given point?

SOLUTION, *by Mr.* LOWRY.

Let A, (fig. 149, pl. 7,) be the given point in the axis AC, and BC a normal to the required curve BH. Draw ED perpendicular to AC, and put AD $= x$, and DB $= y$;

then $\dot{x} : \dot{y} :: y : \frac{y\dot{y}}{\dot{x}} =$ DC, and AC $= x + \frac{y\dot{y}}{\dot{x}}$.

Also, $BC = \surd(DB^2 + DC^2) = \surd(y^2 + \frac{y^2\dot{y}^2}{\dot{x}^2}) = y\surd(1 + \frac{\dot{y}^2}{\dot{x}^2})$;

therefore, if AC have to BC the given ratio of m to n, we have $x + \frac{y\dot{y}}{\dot{x}} = \frac{m}{n}y\surd(1 + \frac{\dot{y}^2}{\dot{x}^2})$ for the fluxionary equation of the curve.

This equation may be integrated in the same manner as that in the solution to question 218, by first taking the fluxion, supposing $\dot{x}$ or $\dot{y}$ to be constant. Thus, if we suppose $\dot{x}$ to be constant, the fluxion of $x + \frac{y\dot{y}}{\dot{x}}$ is $= \dot{x} + \frac{\dot{y}^2}{\dot{x}} + \frac{y\ddot{y}}{\dot{x}}$, and the fluxion of

$$\frac{m}{n} y \sqrt{(1 + \frac{\dot{y}^2}{\dot{x}^2})} = \frac{m\dot{y}}{n\dot{x}\sqrt{(1 + \frac{\dot{y}^2}{\dot{x}^2})}} (\dot{x} + \frac{\dot{y}^2}{\dot{x}} + \frac{y\ddot{y}}{\dot{x}});$$

$$\text{therefore } \dot{x} + \frac{\dot{y}^2}{\dot{x}} + \frac{y\ddot{y}}{\dot{x}} = \frac{m\dot{y}}{n\dot{x}\sqrt{(1 + \frac{\dot{y}^2}{\dot{x}^2})}} (\dot{x} + \frac{\dot{y}^2}{\dot{x}} + \frac{y\ddot{y}}{\dot{x}}),$$

$$\text{or } (\dot{x} + \frac{\dot{y}^2}{\dot{x}} + \frac{y\ddot{y}}{\dot{x}})(1 - \frac{m\dot{y}}{n\dot{x}\sqrt{(1 + \frac{\dot{y}^2}{\dot{x}^2})}}) = 0.$$

From this equation, we have either $\dot{x} + \frac{\dot{y}^2}{\dot{x}} + \frac{y\ddot{y}}{\dot{x}} = 0$, or $1 - \frac{m\dot{y}}{n\dot{x}\sqrt{(1 + \frac{\dot{y}^2}{\dot{x}^2})}} = 0$. In the first case, by integrating, we get $x + \frac{y\dot{y}}{\dot{x}} = a$, where a is a constant arbitrary quantity; therefore $\frac{\dot{y}}{\dot{x}} = \frac{a - x}{y}$, and substituting this value of $\frac{\dot{y}}{\dot{x}}$ in the original fluxionary equation, it becomes $a = \frac{m}{n}\sqrt{\{y^2 + (a - x)^2\}}$, or $\frac{n^2}{m^2} a^2 = y^2 + (a - x)^2$, an equation to the circle whose radius is $\frac{n}{m} a$.

As this equation contains a constant arbitrary quantity, it is, of course, the complete integral. We shall now show how the particular integral is derived from the equation $1 - \frac{m\dot{y}}{n\dot{x}\sqrt{(1 + \frac{\dot{y}^2}{\dot{x}^2})}} = 0$. Here, multiplying by $n\dot{x}\sqrt{(1 - \frac{\dot{y}^2}{\dot{x}^2})}$, and properly reducing the equation, we get $n^2(\dot{x}^2 + \dot{y}^2) = m^2\dot{y}^2$, or $\frac{\dot{y}}{\dot{x}} = \frac{n}{\sqrt{(m^2 - n^2)}}$; and substituting this value of $\frac{\dot{y}}{\dot{x}}$ in the original equation, it becomes

$$x + \frac{yn}{\sqrt{(m^2-n^2)}} = \frac{m}{n}y\sqrt{(1+\frac{n^2}{m^2-n^2})}; \text{ or } x = \frac{y\sqrt{(m^2-n^2)}}{n},$$

an equation to a straight line. So that either a straight line, or a circle, will answer the conditions of the problem. This indeed is obvious, for if a triangle ABC be formed, so that BC may be perpendicular to AB, and AC to CB in the given ratio of m to n; then it is evident that either the straight line AB*b*, which is the locus of the particular integral equation; or any circle described to touch that line, and having its centre on the axis AC, will satisfy the conditions of the problem.

The same otherwise.

Put $z =$ AC; then BC $= \frac{n}{m}z$, $y^2 = \frac{n^2}{m^2}z^2 - (z-x)^2$, and

$y\dot{y} = \frac{n^2}{m^2}z\dot{z} - (z-x)(\dot{z}-\dot{x})$. But $\frac{y\dot{y}}{\dot{x}} =$ DC $= z - x$;

therefore $y\dot{y} = (z-x)\dot{x}$, and $(z-x)\dot{x} = \frac{n^2}{m^2}z\dot{z} - (z-x)(\dot{z}-\dot{x})$,

$$\text{or } \dot{z}\left(\frac{m^2-n^2}{m^2}z - x\right) = 0.$$

From this equation we have $\dot{z} = 0$, or $\frac{m^2-n^2}{m^2}z - x = 0$.

When $\dot{z}$ is $= 0$, z is $=$ to a constant arbitrary quantity a; and writing a for z in the equation $y^2 = \frac{n^2}{m^2}z^2 - (z-x)^2$, it becomes $y^2 = \frac{n^2}{m^2}a^2 - (a-x)^2$,

$$\text{or } y^2 + (a-x)^2 = \frac{n^2}{m^2}a^2, \text{ the same as before.}$$

Again, when $\frac{m^2-n^2}{m^2}z - x = 0$, z is $= \frac{m^2}{m^2-n^2}x$,

and the equation $y^2 - \frac{n^2}{m^2}z^2 - (z-x)^2$ becomes, after proper reduction, $x = \frac{y\sqrt{(m^2-n^2)}}{n}$.

XIX. QUESTION 229, *by Mr.* LOWRY.

Two equal bodies, connected by a flexible string, are placed on a smooth horizontal table, the string being stretched out in a direction parallel to the edge of the table. To the middle of this string, there is fastened another string, at the extremity of which is a given weight that hangs over the edge of the table, in a vertical direction, and, by its descent, communicates motion to the other bodies. Now, suppoſing the weights of the bodies and the length of the connecting string to be given; it is required to define the motion of the descending weight, and to ascertain its place at the end of any assigned time?

SOLUTION, *by Mr.* LOWRY, *the Proposer.*

This question may easily be resolved by the method used by M. D'Alembert, in his *Traite Dynamique*, where the following proposition is assumed as a general principle: "In whatever manner several bodies change their actual motions, if we conceive that the motion which each body would have in the succeeding instant, if it were quite free, is decomposed into two others, of which one is the motion which it really takes in consequence of their mutual action, the second must be such, that if each body were impelled by this force alone, (that is, by the force which produces the second motion,) all the bodies would remain in equilibrio."

Suppose that the bodies A, B are at A′ and B′ (fig. 150, pl. 7.) when the motion commences, and let D′ be the middle of the string. Then the point D′ will be drawn along the straight line D′I, perpendicular to A′B′, by the action of gravity on the descending weight W; and the equal bodies A, B will describe equal portions of the similar curves A′A, B′B with the same velocities. At the end of any indefinite time t, let the bodies on the table be at A and B, and the middle of the string at D; then the distance D′D will be equal to the space through which the weight W has descended in that time. Let the velocity of the bodies in the curves A′A, B′B $= u$, and their mass, or weight, $= w$. And the velocity of the descending weight $= v$, and its mass, or weight, $=$ W. Draw AE perpendicular to D′D; and put $a =$ AD $=$ A′D′ $=$ half the length of the string, $s =$ D′D, $x =$ D′E, $y =$ EA, $r =$ the arch A′A, $z =$ ED, and $g = 32\frac{1}{6}$ feet, the measure of the accelerative force of gravity.

Then, at the end of the time t, when the bodies on the table have acquired the velocity u, if we suppose the weight W to be disengaged from the string, so that the bodies may be at liberty to move freely, it is evident that the bodies A, B would proceed with the uniform velocity u; and in the succeeding instant $\dot{t}$ the descending body W, would acquire the velocity $v + g\dot{t}$, and the quantities of motion would then be

$$wu,\ wu \text{ and } W\ (v + g\dot{t});$$

or $w\ (u + \dot{u} - \dot{u})$, $w\ (u + \dot{u} - \dot{u})$ and $W\ (v + \dot{v} + g\dot{t} - \dot{v})$.

But the motions which the bodies really take at that time, are $w\ (u + \dot{u})$, $w\ (u + \dot{u})$, and $W\ (v + \dot{v})$; therefore the motions

$w\ (-\dot{u})$, $w\ (-\dot{u})$, and $W\ (g\dot{t} - \dot{v})$ must be such that if the bodies were impressed with these motions alone, the system would be in equilibrio;

that is, $2w\ (-\dot{u})$ must be $= \frac{v}{u} W\ (g\dot{t} - \dot{v})$;

or $Wg\dot{t} = W\dot{v} + 2w\frac{u\dot{u}}{v}$. But $\dot{t}$ is $= \frac{\dot{s}}{v}$, therefore $gW\dot{s} = Wv\dot{v} + 2wu\dot{u}$; and taking the fluents, we get

$$2gWs = Wv^2 + 2wu^2 \ \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

But it is evident that the string AD is a tangent to the curve at A, therefore $\dot{r} : \dot{x} :: a : z$, or $\dot{r} = \frac{a\dot{x}}{z}$. Also $\dot{x} : -\dot{y} :: z : y$, or $-\dot{y} = \frac{y\dot{x}}{z}$; and since $a^2 - y^2$ is $= z^2$, $-\dot{y}$ is $= \frac{z\dot{z}}{y}$; therefore $\frac{y\dot{x}}{z}$ is $= \frac{z\dot{z}}{y}$, $\dot{z} = \frac{y^2\dot{x}}{z^2}$ and $\dot{x} = \frac{z^2\dot{z}}{a^2 - z^2}$; consequently $\dot{s} = \dot{x} + \dot{z} = \dot{x} + \frac{y^2\dot{x}}{z^2} = \frac{a^2\dot{x}}{z^2} = \frac{a^2\dot{z}}{a^2 - z^2}$; $s = \int \frac{a^2\dot{z}}{a^2 - z^2}$ $= a \times \log. \frac{a + z}{a - z}$, and $\frac{v}{u} = \frac{\dot{s}}{\dot{r}} = \frac{z}{a}$, or $u^2 = \frac{a^2v^2}{z^2}$.

This value of u^2 being substituted in the equation (1), it becomes $2gWs = Wv^2 + 2w\frac{a^2v^2}{z^2}$; from whence we deduce

$$v^2 = \frac{2gWsz^2}{Wz^2 + 2wa^2} = \frac{2gWa \log. \frac{a + z}{a - z}}{Wz^2 + 2wa^2}; \text{ and}$$

$$t = \int \dot{s} \sqrt{\left(\frac{Wz^2 + 2wa^2}{2gWsz^2}\right)} = \int \frac{a^2\dot{z}}{a^2 - z^2} \sqrt{\left(\frac{W^2z^2 + 2wa^2}{2gWz^2 a \log. \frac{a + z}{a - z}}\right)}$$

In the particular case when $z = a$, we have

$$v^2 = \frac{2gW}{W + 2w}, \text{ and } t = s\sqrt{\left(\frac{W + 2w}{2gW}\right)} \text{ as they ought to be.}$$

To find the equation of the curve described by the bodies, we have

$$\dot{x} = \frac{z^2\dot{z}}{a^2 - z^2} = \frac{\dot{y}}{y}\sqrt{(a^2 - y^2)}, \text{ and}$$

$$x = a \times \log. \frac{a + \sqrt{(a^2 - y^2)}}{y} - \sqrt{(a^2 - y^2)}.$$

XX. QUESTION 230, *by Mr.* Lowry.

A circle and a sphere being given by poſition, it is required to determine the perspective representation of the circle on the surface of the sphere, and to find the rectification of the curve; ſuppoſing the eye to be at the centre of the sphere?

Solution, *by Mr.* Lowry, *the Propoſer.*

Let C (fig. 151, pl. 7.) be the centre of the sphere, PQR the circle given by position, and O its centre. Draw CG perpendicular to the plane of the circle; then G is a given point, and CG a given line. Draw GO to meet the circle in Q and R, and let P be a point in the periphery of the circle; then if PC be drawn to meet the surface of the sphere in p, it is evident that the point p is in the required curve. To find the position of this point on the surface of the sphere, let GP, Gp be drawn. Then since CG is perpendicular to the plane GPR, the angle CGP is a right-angle, therefore if we make a right-angled triangle CGP, and take Cp = to the radius of the sphere, the distances GP, Pp will be known: and if with these distances and the centres G, P we describe arches on the surface of the sphere, they will intersect in p. In this manner we may find as many points on the surface of the sphere as we please, and the curve described through these points will be the representation of the given circle.

If we conceive the curve to be projected on the plane which passes through the points C, G, O, we may easily describe the projected curve. For if pm be drawn perpendicular to the plane CGO, the point m will be in the curve. Draw PM perpendicular, and mg parallel to RQ, and join p, g Then it is evident that the triangles GPM, gpm are similar, therefore PC :

$pC = r' :: PM : pm :: GM : gm :: CG : Cg$, and the rectangular co-ordinates gC, gm are equal to $r' \times \frac{CG}{PC}$ and $r' \times \frac{GM}{PC}$ respectively.

The nature of the curve may also be defined by equations as follows: put $x = Cg$, $y = gm$, $z = pm$, $d = GO$, $h = CG$, and $r = OP$; then $x : y :: h : GM = \frac{yh}{x}$; therefore $OM = \frac{yh}{x} - d$, and $PM^2 = r^2 - (\frac{yh}{x} - d)^2$.

Also $h^2 : x^2 :: r^2 - (\frac{yh}{x} - d)^2 : z^2$,

$$\text{or } z^2 = \frac{x^2}{h^2}\left\{r^2 - (\frac{yh}{x} - d)^2\right\} \ldots\ldots\ldots\ldots\ldots (1)$$

And, by the property of the sphere,

$$z^2 + y^2 + x^2 = r'^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

These equations serve to define the nature of the curve on the surface of the sphere; and if we eliminate z, we get $\frac{x^2}{h^2}\left\{r^2 - (\frac{yh}{x} - d)^2\right\} = r'^2 - x^2 - y^2$, for the equation of the curve when projected on the plane OGC. By reducing this equation it becomes $(r^2 + h^2 - d^2)\, x^2 - h^2r'^2 + 2dxyh = 0$, which is an equation to the common hyperbola; and when d is $= 0$, x is $= \frac{h^2r'^2}{\sqrt{(r^2 + h^2)}}$, and the projection of the curve is then a straight line.

Again, if x be eliminated from the equations (1), (2), we have

$$\sqrt{(r'^2 - y^2 - x^2)} = \frac{dhy}{r^2 - d^2} + \frac{hr}{r^2 - d^2} \sqrt{(y^2 - z^2 - \frac{d^2z^2}{r^2})}$$

for the equation of the curve when projected on the plane GPR. When d is $= 0$, $(y^2 + z^2)\,(h^2 + r^2)$ is $= r'^2r^2$, an equation to a circle.

From the above equations we may find the fluxions of x, y, and z, in terms of one of these variables and its fluxion, and then $\int \sqrt{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}$ will express the length of the curve: but the operation will be much simpler if we proceed in the following manner: If the circle PRQ be considered as the base, and the point C as the vertex of a cone, it is obvious that the curve in the question is formed by the mutual intersection of the cone and spherical superficies; and if we conceive the surface of the cone to be extended on a plane, (fig, 152, pl. 7.) and the arch EF described about the centre C, with a distance equal to the radius of the sphere, it is evident that the arch EF is the developement of the curve on the plane, and consequently its length is equal to the length of the curve.

Let w be any variable arch PR, and PP′ a small part of the arch. Draw PC, P′C to meet the arch EF in p and n, and draw PN perpendicular to CP′. Let φ be the measure of the arch w on the circle having unity for its radius; put u = the arch Ep, and p = CP. Then by similar triangles $\dot{u} = pn$: PN :: Cp : CP, or $\dot{u} = \frac{r'}{p} \times$ PN. But PN is $= \sqrt{(\text{P}'\text{P}^2 - \text{P}'\text{N}^2)}$ $= \sqrt{(\dot{w}^2 - \dot{p}^2)}$, therefore $\dot{u} = \frac{r'}{p} \sqrt{(\dot{w}^2 - \dot{p}^2)}$.

Now it is evident that $p^2 = h^2 + d^2 + r^2 - 2rd \cos \varphi$, hence $\dot{p} = r\dot{\varphi} \times \frac{d \sin \varphi}{p}$. But $\dot{w} = r\dot{\varphi}$; therefore, by substitution,

$$\dot{u} = \frac{r'r\dot{\varphi}}{p} \sqrt{(1 - \frac{2d \sin^2 \varphi}{p})} = r'r\dot{\varphi} \sqrt{\frac{\{h^2 + (r - d \cos \varphi)^2\}}{h^2 + d^2 + r^2 - 2rd \cos \varphi}},$$

and the length of the curve

$$= r'r \int \dot{\varphi} \frac{\sqrt{\{h^2 + (r - d \cos \varphi)^2\}}}{h^2 + d^2 + r^2 - 2rd \cos \varphi}.$$

For the method of integrating this expression, we shall refer to Art. 54, Memoire sur les Transcendental Elliptiques, par Le Gendre.

It is obvious that most of the properties noticed in the solutions to questions 220 and 231, will be true for the curve in the question, as well as for a small circle.

XXI. QUESTION 231, *by Mr.* LOWRY.

If from two given points on the ſurface of a ſphere, two great circles be drawn to any point D in the circumference of a ſmall circle given by poſition, meeting it again in E and F: then the great circle joining the points E, F will either paſs through a given point P; or a great circle drawn through E and P will cut off from the ſmall circle EDF, an arch FG of a given length.

SOLUTION,

SOLUTION, *by Mr.* LOWRY.

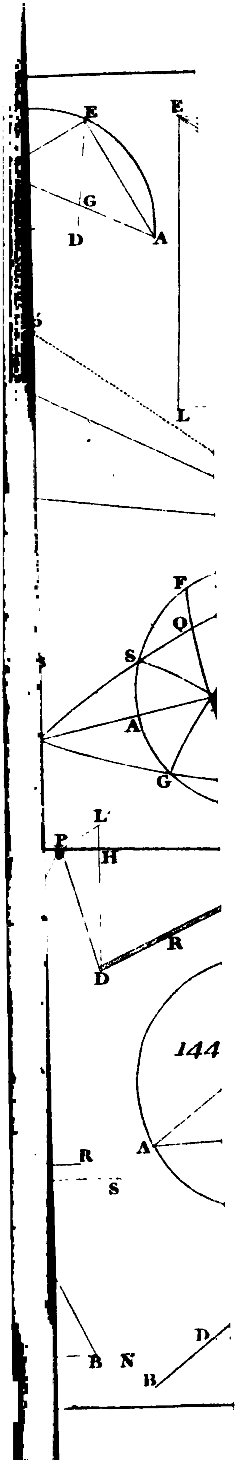
Let A, B (fig. 153, pl. 8.) be the given points on the surface of the sphere, and let the planes of the great circles AED, BFD, AB and GPE meet the plane of the small circle in the straight lines *a*ED, *b*FD, *ab* and G*p*E; then it is evident that *a* and *b* are given points.

Also, let the straight line drawn from P, to the centre O of the sphere, meet the line GE in *p*; and through *p*, and the centre C of the given circle, draw a straight line to meet *ab* in I, and the circle in L and M; then, since (by hyp.) P is a given point, *p* will also be a given point; and because the arch EF either passes through P, or the arch FG is of a given length, the line EF must either pass through *p*, or the angle FE*p* must be a given angle; therefore the point *p* must be so situated, that straight lines being drawn from the points *a*, *b* to any point D in the periphery of the circle, meeting it again in E and F, the line EF may either pass through *p*, or make a given angle with a straight line passing through that point. This we know (from the solution to quest. 210) will be the case when the rectangle *a*I × *ab* is equal to the square of the tangent *a*H drawn to the circle from *a*; and when the line IM is harmonically divided in the points I, L, *p*, M; or when IC × C*p* is $= CL^2$. For then the angle FE*p* is equal to the difference between the given angle *b*IC and a right-angle; and when *b*IC is a right-angle, the line EF will pass through *p*.

Let the plane which passes through the points I, O, C meet the surface of the sphere in the great circle QLPM, and the plane *b*O*a* in the straight line IQO. Also let *b*K be drawn to touch the circle at K, and through the points A, H; and B, K let arches of great circles be drawn; and join O, H and O, K. Then since the planes O*a*H, O*b*K touch the small circle at H and K, the arches AH, BK will also touch the circle in these points, and are therefore given arches; and it is evident that *a*H is the tangent, and *a*O the secant, of the arch AH; and *b*K the tangent, and *b*O the secant, of the arch BK.

Now take A′B′ = the given arch AB, and make A′O′, B′O′ each equal to the radius of the sphere. Take O′*a*′ = to the secant of the arch AH, and O′*b*′ = to the secant of the arch BK, and join *a*′ *b*′. Also make the rectangle I′*a*′ × *a*′*b*′ = tan.^{2} AH, and draw I′O′ meeting the arch A′B′ in Q′. Then the arch A′Q′ is evidently equal to the arch AQ.

Again, since the line IM is harmonically divided in the points I, L, *p*, M;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I, L, *p*, M; the arch QM is also harmonically divided in the points Q, L, P, M (solution to quest. 220); or which is the same thing, tan. QS tan. PS is = tan.^{2} SL, S being the pole of the circle EDF.

Hence, if on the arch AB, we take AQ = to the arch A′Q′, and the arch SQ be divided in P, so that tan. QS tan. PS may be equal tan.^{2} SL, the point P will be that required.

In the figure to this solution, the given points A, B are taken without the circle EDF; but the same reasoning will apply when they are within the circle; or when one is within, and the other without; only, when they are within the circle, the lines aH, bK must be drawn perpendicular to the diameters which pass through the points a, b; and the distances aH, aO, bK and bO are the sines and cosines of the arches AH, BK instead of their tangents and secants.

It is evident (from the solution to quest. 210, and what has been done above,) that the proposition may be extended to any number of given points A, B, C, D, &c.; and it is also obvious from thence how a great circle spherical polygon may be inscribed in a given small circle, so that each side may pass through a given point on the surface of the sphere.

Scholium. We have already remarked that, when the line IC is at right angles to ab, the line EF will pass through the point p. In this case the sum of the squares of aH and bK is equal to the square of ab; and if the line aF meet the circle again in N, the points b, N, E are in a straight line; or if the opposite sides ED, FN of a trapezium inscribed in the circle EDF, intersect in a, the sides FD, EN will intersect in b. Also the diagonals DN, EF will intersect in the given point p. These properties are generally known when the figure is described on a plane (see solution to quest. 240, Old Series of the Repository), and it is obvious that there are similar properties when the figure is on the surface of a sphere. Thus, if the arch AF meet the small circle in N, the points B, N, E are in the arch of a great circle; or if the opposite sides of any great circle trapezium, inscribed in the small circle EDF, intersect in A, the other opposite sides will intersect in B. Also the great circles joining the points E, F and D, N will intersect in the given point P.

Again, let r be the radius of the sphere, then by trigonometry

$$ab^2 = aO^2 + Ob^2 - 2aO \times Ob \times \cos aOb$$

$$= \sec^2 AH + \sec^2 BK - \frac{2 \sec AH \sec BK \cos AB}{r},$$

and $aH^2 + bK^2 = \tan^2 AH + \tan^2 BK = \sec^2 AH + \sec^2 BK - 2r^2$;

therefore, when ab^2 is $= aH^2 + bK^2$,

$$\frac{\sec AH \sec BK \cdot \cos AB}{r} = r^2$$

$$\text{or, } r^2 \cos AB = \frac{r^4}{\sec AH \sec BK} = \cos AH \cos BK.$$

From whence we have the following properties.

1st. Let EDF (fig. 154, pl. 8.) be a small circle given by position, and A and B two given points on the surface of a sphere, such, that the rectangle of the radius of the sphere, and the cosine of AB, may be equal to the rectangle of the cosines of the tangents AH, BK drawn to the circle from A and B, and let great circles be drawn from the points A and B to any point D in the periphery of the circle EDF, meeting it again in E and F: then the arches joining the points A, F and B, E will intersect in the periphery of the circle EDF. Also the arches joining the points E, F and D, N will intersect in a given point P.

2d. If the opposite sides of a spherical trapezium inscribed in a small circle of the sphere, be produced to meet; the rectangle of the radius of the sphere, and the cosine of the arch joining the points of concourse, will be equal to the rectangle of the cosines of the tangents drawn to the circle from these points.

XXII. QUESTION 292, *by Mr.* WALLACE.

Let F be the focus of a conic ſection, and P a given point in the axis. Draw EA to any point of the curve; take PB in the axis equal to FA and join AB. There is a ſtraight line given by poſition, which divides AB into parts having to each other a given ratio. Required the demonſtration?

FIRST SOLUTION, *by Mr.* LOWRY.

Suppose that LI (fig. 155, pl. 8.) is the straight line given by position which divides AB into parts (AI, IB) having a given ratio.

Let AE, drawn parallel to the axis, meet PI produced in E; then, by similar triangles, AI is to IB as AE to BP, or AF; but AI has to IB a given ratio by hypothesis; therefore AE has to AF a given ratio; and, consequently, the point E is in the directrix of the conic section. Let the directrix ED be drawn to meet the axis in D, and let V be the vertex of the axis. Then since EI has to IP the given ratio of AE to AP (PB); that is, of DV to VF; and the point E being always in the directrix ED, the point I will be in a straight line LI, given by position, and parallel to ED; wherefore DL is to LP as DV to VF. Hence, if DP be divided at L, in the given ratio of DV to VF, the point L will be given, and the straight line LI, which divides AB in a given ratio, will be given by position.

The proposition, however, is not confined to the particular case where PB is taken on the axis, but is true when PB is taken on any straight line given by position.

For, if P (fig. 156, pl. 8.) be a given point in a straight line PQ given by position, (which meets the curve at V,) and BP be taken = FA; there is a straight line LI, given by position, which divides AB into parts having a given ratio. Let AE, drawn parallel to PQ, meet PI produced in E; then it may be shewn in the very same manner as above, that the point E is in the directrix of the conic section; and that DL is to LP as DV to VF; therefore L will be a given point, and the line LI will be given by position, and AB will be divided at I in the given ratio of DV to VF.

SECOND SOLUTION, *by Mr.* CUNLIFFE.

Let VAF (fig. 155, pl. 8.) be a given conic section, F its focus, V its principal vertex, DE its directrix at right angles to DVF. Draw AE parallel to DF meeting DE in E, and draw PE intersecting AB in I. Lastly, through I draw LM parallel to DE, meeting DF in L and EA in M.

By the known property of the directrix FA = PB : EA :: FV : VD, a given ratio. Also, per the similar triangles, EIA, PIB, and the parallels LI, DE,
PB : EA :: IB : IA :: PI : IE :: PL : LD; therefore
IB : IA :: PL : LD :: FV : VD, a given ratio. Wherefore divide PD in L in the given ratio of FV to VD, and draw LM parallel to the directrix; then LM will be a right line given by position, which always divides AB in a given ratio.

XXIII. QUESTION 233.

To this question we have received no answer.

XXIV. QUESTION 234, *by Mr.* IVORY.

A point may be found within the baſe of every oblique cone, ſuch, that if any plane be drawn through the vertex and the point found to cut the cone, then the vertical angle of the

triangle

triangle, which is the common ſection of the plane and the con will be biſected by the right line drawn from the point found to the vertex of the cone ?

SOLUTION, *by Mr.* LOWRY.

LEMMA. Let a straight line EI (fig. 157, pl. 8.) be harmonically divided in the points I, F, P, E; and from the points of division let straight lines be drawn to any point V, so that VI may be perpendicular to PV; then the angle EVF is bisected by the line PV.

For through P, draw GPH parallel to IV, meeting VF produced in H; then by similar triangles

FI : PF :: IV : PH,

and PE : IE :: PG : IV;

but, by hypothesis, IF : IE :: PF : PE;

therefore, ex æquo, IV : PH :: IV : PG;

therefore PH is equal to PG, and the right-angled triangles PGH, PHV are equal in every respect; consequently the angles GVP, HVP are equal, or the angle EVF is bisected by the line PV.

Now let a plane be drawn through the axis VC (fig. 158, pl. 8.) of the cone, perpendicular to the plane of the base, meeting the surface of the cone in the straight lines AV, BV. Draw VP to bisect the angle AVB; then P is the point required. That is to say, if any plane be drawn through PV, to meet the surface of the cone in the straight lines EV, FV, then the angle EVF will be bisected by the line PV.

Let a plane be drawn through V, perpendicular to the line VP, meeting the plane of the base in the straight line HI, and the diameter AB produced in D; and join VD. Then VD is perpendicular to VP (Euc. XI. 17.) and IDH perpendicular to AD (XI. 18.). But since PV bisects the angle AVB, and VD is perpendicular to PV, it is evident that VD bisects the exterior angle BVa; therefore AP : PB :: AV : BV :: AD : BD (VI. 3. A); or the diameter is harmonically divided in the points D, B, P, A.

Let DF be produced to meet DH in I, and join VI. Then because the diameter of the circle is harmonically divided in the points D, B, P and A; and the line HDI is perpendicular to AD; any line EI drawn through P to meet the circle in E and F, and the straight line DI in I, is harmonically divided in the points I, F, P and E. This property of the circle is generally known. Moreover the plane DIV being perpendicular to VP, the line IV is perpendicular to PV; therefore, by the lemma, the angle EVF is bisected by the line PV.

When EF is parallel to DI, it is obvious that the angles FPV, EPV are right-angles, and that EP is equal to PF; therefore, in this case, the triangles EPV, FPV are equal in every respect; consequently the line PV bisects the angle EVF.

Corol. The rectangle EV × VF is equal to the constant rectangle AV × VB.

For $EV \times VF = VP^2 + EP \times PF = VP^2 + AP \times PB = AV \times VB$. (VI. Prop. B.)

XXV. QUESTION 235, *by Mr.* IVORY.

It is required to ſhew that the ſurface of a right cone on an elliptical baſe may be found by the rectification of the ellipſe.

By a right cone on an elliptical baſe is meant ſuch a cone as has an ellipſe for its baſe, and likewiſe the right line, drawn from the vertex to the centre of the ellipſe, perpendicular to the plane of the baſe?

SOLUTION, *by Mr.* LOWRY.

Let V (fig. 159, pl. 8.) be the vertex, C the centre of the base, CB the semi-transverse, and CD the semi-conjugate axis. Let $DH = z$ be a variable arch of the elliptic base, and H*h* an indefinitely small arch $= \dot{z}$. Join VH, V*h* and CH; and draw HI perpendicular to CB. Put $h =$ CV, the height of the cone; $e =$ the excentricity of the ellipse, the semi-transverse being 1; $r =$ VH, the distance of the point H from the vertex $x =$ CI and $y =$ IH.

From *h* draw a line perpendicular to VH; then this perpendicular will be one leg of a right-angled triangle, of which $\dot{z}$ is the hypothenuse, and $\dot{r}$ the other leg; it will therefore be equal to $\sqrt{(\dot{z}^2 + \dot{r}^2)}$; and the fluxion of the conical surface DVH, or the area of the small triangle HV*h*, will be equal to $\frac{1}{2}r\sqrt{(\dot{z}^2 + \dot{r}^2)}$. Therefore, if *s* be put for the surface of the cone we have $\dot{s} = \frac{1}{2}r\sqrt{(\dot{z}^2 + \dot{r}^2)}$.

The equation of the ellipse being $y^2 = (1 - e^2)(1 - x^2)$, we have $CH^2 = x^2 + y^2 = x^2 + (1 - e^2)(1 - x^2) = 1 - e^2 + e^2x^2$; therefore $r^2 = CV^2 + CH^2 = h^2 + 1 - e^2 + e^2x^2$, and $\dot{r} = e^2x\dot{x} \div r$. Also $\dot{z}$ is $= \dot{x}\sqrt{(\frac{1 - e^2x^2}{1 - x^2})}$; therefore, by substitution,

$\dot{s} = \frac{r}{2}\dot{x}\sqrt{(\frac{1-e^2x^2}{1-x^2} + \frac{e^4x^2}{r^2})}$; and putting for r its value $\sqrt{(h^2+1-e^2+e^2x^2)}$, and reducing the quantity under the radical, we get

$$\dot{s} = \tfrac{1}{2}\dot{x}\sqrt{(\frac{h^2+1-e^2+h^2e^2x^2}{1-x^2})}.$$

But since $1-e^2$ is equal to the square of CD, h^2+1-e^2 is equal to the square of VD, the line drawn from the vertex to the extremity of the conjugate axis; therefore, if d be put for that line, we have

$$\tfrac{1}{2}d\dot{x}\sqrt{(\frac{1-\frac{h^2e^2}{d^2}x^2}{1-x^2})}$$ for the fluxion of the conical surface; the fluent of which is obviously $\frac{1}{2}d$ multiplied by the periphery of the ellipse whose excentricity is $(h^2 \div d^2)e^2$, and semi-transverse 1.

The same otherwise.

Fig. 160, pl. 8. Let a plane, parallel to the plane of the base, be drawn through any point E, in the line VC, to meet the plane CVB in the straight line EQ. Draw FG perpendicular to EQ meeting the surface of the cone in G, and VF, VG to meet the plane of the base in I and H; then the point H will be in the periphery of the base, and HI will be perpendicular to CB. Put y = VE, x = EF, z = FG, and s = the conical surface bounded by the plane EQG; then,

$$s = \int\int \dot{x}\dot{y}\sqrt{(1+\frac{\dot{z}^2}{\dot{x}^2}+\frac{\dot{z}^2}{\dot{y}^2})}^*.$$

Put h = VC, u = CI and v = IH; then by similar triangles

$y : x :: h : u = hx \div y$, and $y : z :: h : v = hz \div y$;

$$\text{hence } x = \frac{uy}{h} \text{ and } z = \frac{vy}{h};$$

Take fluxions, supposing y constant; then

$$\dot{x} = \frac{y}{h}\dot{u}, \text{ and } \dot{z} = \frac{y}{h}\dot{v}; \text{ therefore } \frac{\dot{z}}{\dot{x}} = \frac{\dot{v}}{\dot{u}}.$$

Again, $\dot{z} = \frac{v\dot{y}}{h} + \frac{y\dot{v}}{h}$, and $\frac{\dot{z}}{\dot{y}} = \frac{v}{h} + \frac{y\dot{v}}{\dot{y}h}$; but $\dot{y}$ is $= -y\frac{\dot{u}}{u}$, when x is constant; therefore

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* For the investigation of this formula, see Traité E'lémentaire de calcul def. et integ. par La Croix. Art. 251.

$\dot{z} \div \dot{y}$

$$\frac{\dot{z}}{\dot{y}} = \frac{v}{h} - \frac{u\dot{v}}{h\dot{u}}.$$

Whence, by substitution,

$$s = \frac{1}{h^2} \int\!\!\int y\dot{y}\dot{u} \sqrt{\left\{h^2 + h^2\left(\frac{\dot{v}}{\dot{u}}\right)^2 + \left(v - \frac{u\dot{v}}{\dot{u}}\right)^2\right\}}.$$

The fluent relative to y, being taken from $y = 0$ to $y = h$, we have $s = \frac{1}{2}\int \dot{u} \sqrt{\left\{h^2 + h^2\left(\frac{\dot{v}}{\dot{u}}\right)^2 + \left(v - u\frac{\dot{v}}{\dot{u}}\right)^2\right\}}$ for the surface of the whole cone.

This is a general expression for the surface of a cone, let the figure of the base bo what it may: In the present case $v^2 = (1 - e^2)(1 - u^2)$; therefore

$$v = \sqrt{(1-e^2)(1-u^2)},\ \frac{\dot{v}}{\dot{u}} = -u\sqrt{\left(\frac{1-e^2}{1-u^2}\right)},\ \text{and}\ \left(v - \frac{\dot{v}}{\dot{u}}u\right)^2 = \frac{1-e^2}{1-u^2};$$

therefore, by substitution, and proper reduction, we get $s = \frac{1}{2}\int \dot{u} \sqrt{\left(\frac{h^2 + 1 - e^2 - h^2e^2u^2}{1 - u^2}\right)}$, the same expression as before.

XXVI. QUESTION 236, *by Mr.* IVORY.

From a point aſſumed without a cone, let two tangents be drawn to the same conic surface, or to the opposite surfaces; and let a plane be drawn through the vertex of the cone and the two points of contact of the tangents: then, if a right line be draw from the aſſumed point to cut the plane, and either of the conic ſurfaces in two points, that right line will be harmonically divided?

N. B. The demonstration of some properties of all the conic sections readily follows from this proposition.

SOLUTION, *by Mr.* LOWRY.

Let P (fig. 161, pl. 8.) be the assumed point, PQ and PR the tangents; and let the plane drawn through the vertex, and the points of contact P and Q, cut the conic surface in the lines VQE and VRF, and the plane drawn through P, parallel to the plane of the base, in the line EF. Then any straight line drawn from P to meet the plane VEF in I, and either of the conic surfaces in H and G, is harmonically divided in the points P, H, I and G. For let a plane be drawn through the vertex V and the line PG to cut the conic surface in the lines VGA and VHB, the plane VEF in the line VD, and the plane of the circle in the line ADBP; also draw PE and PF. Then since the lines PQ and PR touch the conic surface, or the opposite surfaces, at Q and R, the planes VQEP and VRFP will touch the conic surface in the lines VE and VF, and consequently PE and PF will touch the circle AEBF at E and F; therefore (by a well known property of the circle) PA is harmonically divided in the points P, B, D and A; therefore the lines PV, BV, DV and AV are harmonicals, and since PG meets these lines in the points P, H, I and G, it is harmonically divided in these points, (Part 2nd. Lem. 1. B. V. Hamilton).

If PG be parallel to VA, or to VB, then PI is bisected in H, or in G, (Part 1st. Ibid.)

Cor. 1. A line drawn through any point in VP to meet the plane VEF, and either of the conic surfaces, is harmonically divided. For it will meet four harmonicals.

Cor. 2, If GH be bisected in K; then (by reason of the harmonical division) KI × KP is equal to KH^2 or KG^2.

It follows from this proposition, that if two tangents be drawn from the same point to any conic section, then any line drawn through P to meet the section in H and G, and the line RQ that joins the points of contact in I, is harmonically divided in the points P, H, I and G.

If the assumed point be within the cone, we have the following proposition, which is analogous to the preceding one.

Through any assumed point P within a cone, (fig. 162, pl. 8.) let a plane be drawn parallel to the plane of the base, cutting the conic surface in the circular section ABL. Let C be the centre of the circle, and on CP let the point Q be taken such, that CQ may be a third proportional to CP, and the radius CL, and let EF be drawn perpendicular to CQ. Then any right-line drawn through P to meet the conic surface in G and H, and the plane VEF in I, is harmonically divided in the points G, P, H and I.

For, let the plane drawn through V, and the line GPI, cut the conic surface in the lines VAG and VHB, the plane of the circle in the line APBD, and the plane VEF in the line VID. Also let PV be drawn. Then because CL, the radius of the circle, is a mean proportional between CP and CQ, and EF is perpendicular to CQ, therefore (by a known property of the circle) the line DA is harmonically divided in the points A, P, B and D; therefore the lines GAV, PV, BV and DV are harmonicals, and consequently the line GI is harmonically divided in the points G, P, H and I.

If GPI be parallel to VA, or to VB, then PI is bisected in H, or in G; and if it be parallel to the plane VEF, then GH is bisected in P.

Cor. 1. A line drawn through any point in VD to meet the plane VEF, and the conic surface, is harmonically divided.

Cor. 2. If GH be bisected in K, then (by reason of the harmonical divison) KP × KI is equal to KH^2 or KG^2.

Cor. 3. If any plane MN be drawn through P to meet the plane VEF in the right-line IO, and from any point I in this line, tangents be drawn to the conic surface at M and N, the line MN that joins the points of contact, will pass through P. For let IP be drawn to meet the conic surface in H and G, and if MN does not pass through P, let it meet IG in p: then, by what has been already proved, IG will be harmonically divided in the points I, H, P and G, and also in the points I, H, p and G, which is absurd; therefore MN passes through P.

Cor. 4. When MN is parallel to IO, it is bisected in P.

From what has been demonstrated it follows:

That, if on the diameter GH of a conic section, two points P and I be assumed such, that PK × KI may be equal to the square of the semi-diameter KH (or in the parabola that PH may be equal to HI), and IO be drawn parallel to the line MN that joins the points of contact of tangents drawn to the section from I: then any right-line drawn through P to meet the conic section in R and S, and the line IO in T, is harmonically divided in the points R, S, P and T.

Dr. O'Riordan *answered this question.*

XXVII. QUESTION 237, *by Mr.* Ivory.

It is required to determine the poſition of the axes of the repreſentation of a circle given by poſition in a horizontal plane: ſuppoſing the place of the eye to be given, and the plane of the picture perpendicular to the plane of the horizon?

Solution, *by Mr.* Lowry.

Lemma 1. Let a cone be cut by a plane, PQ (fig. 163, pl. 8.), which neither passes through the vertex, nor is parallel to the plane of the base; and let a plane, as VMN, be drawn through the vertex, parallel to the cutting plane, to meet the plane of the base in the line MN. From the centre O, draw OD perpendicular to MN, and take OE a third proportional to OD, and the radius OA, or OB, of the circle. From any point I in the line MN, let tangents IG and IH be drawn to the circle, and through the vertex V, and the line GH that joins the points of contact, let a plane be drawn to cut the conic surface in the lines GRV and HSV, and the cutting plane in the line RS. Draw VI; then any line *pq*, parallel to VI, and terminated by the conic surface, is bisected by the line RS.

Because VI and *pq* are parallel, they are contained in one plane. Let that plane be drawn to meet the conic surface in the lines V*p*T and V*q*K, the plane of the base in the line IKLT, and the plane VGH in the line V*s*L. Then because CB, or CA, the radius of the base of the cone, is a mean proportional between CE and CD, and MN is perpendicular to CD; therefore the line GH that joins the points of contact of tangents drawn to the circle from any point in MN, will pass through E, and the line IT will be harmonically divided in the points I, K, L, and T; therefore VI, VK, VL, and VT are harmonicals, and since *pq* meets three of these harmonicals, and is parallel to the fourth, it is bisected in the point *s* (see the lem. referred to in the solution to the preceding question).

Lemma 2. Let the point E (fig. 164, pl. 8.) be found as before; and let the points I and K be taken in the line MN, such, that the lines IE and KE being drawn to meet the circle in the points W and T, and H and G, the tangents drawn to the circle at W and T, may intersect in K; and those drawn to the circle at G and H, may intersect in I. Let the plane drawn through V, and the line IT, meet the conic surface in the lines VT and VW, the cutting plane in the line PCQ, and the plane VMN in the line VI. Also let the plane drawn through V and the line KG meet the conic surface in the lines VG and VH, the cutting plane in the line RCS, and the plane VMN in the line VK. Then PQ and RS are conjugate diameters of the section PSQR.

Because PQ and VI are the common sections of the plane VTI and two parallel planes, (viz. the plane of the section and the plane VMN), therefore PQ is parallel to VI; therefore (*lem.* 1.) all the lines in the cutting plane that are parallel to PQ or VI, and terminated by the section, are bisected by RS.

Again, because RS and VK are the common sections of the plane VGK and two parallel planes, (viz. the plane of the section and the plane VMN), therefore RS is parallel to VK, and therefore (lem. 1.) all the lines parallel to RS, or VK and terminated by the section, are bisected by the line PQ.

Hence it is evident that PQ and RS are conjugate diameters of the section PSQR.

Cor. 1. Because of the parallels the angle IVK is equal to the angle SCQ, made by the conjugate diameters; therefore when IVK is a right-angle, PQ and RS are the axes of the section.

Lemma. 3. Prob. To determine the points I and K for any given cone, so that the angle IVK may be a right-angle.

Let VZ be drawn perpendicular to IK, and conceive the triangle IVK to be turned down till it coincides with the plane of the base, and through the points I, V, K describe a circle to meet the line OD in L. Join IL and KL, and draw IO to meet GH in X; then ILK is a right-angle, and because GH joins the points of contact of tangents from I, the line IO drawn to the centre is perpendicular to GH; therefore the right-angled triangle OEX is similar to the triangle EKD, and also to the triangle ODI; therefore the triangles EKD and ODI are similar, and $ID : OD :: ED : DK$; therefore $ID \times DK =$ ($DL^2 =$) $OD \times ED =$ a given space; therefore DL is a given line, and L is a given point. And since V is also a given point, it is manifest that, if VL be bisected perpendicularly by a line meeting IK in Y, then Y will be the centre, and YL, or YV the radius of the circle ILVK.

Cor. 1. Let a plane be drawn through V, parallel to the plane of the base, meeting the cutting plane in the line *mn*; and let the plane VBAD meet the cutting plane in the line DC*d*, and the plane V*mn* in the line V*d*, and draw VO, and VL to meet D*d* in *o* and *l*. Also through V and the lines IL and KL let planes be drawn to meet the cutting plane in the lines *il*, *kl*, and the plane V*mn* in the lines *i*V and *k*V. Then (because of the parallel planes PSQR and VIK) *il* is parallel to IV, or PQ, and *kl* is parallel to VK, or RS; therefore the angle *kli* is equal to the angle IVK ($= iVk$), or a right-angle.

Cor. 2. *dl* is a mean proportional between *do* and *d*C. For, by similar triangles,

$$OD : DV :: dV : do, \text{ and } DE : DV :: dV : dC;$$

$$\text{therefore } OD \times DE = DL^2 : DV^2 :: dV^2 : do \times dC.$$

$$\text{But } DL^2 : DV^2 :: dV^2 : dl^2; \text{ therefore } do \times dC = dl^2.$$

To apply what has been demonstrated in the preceding lemmas to the solution of the question, let THWG (fig. 165, pl. 8.) be

the

the original circle, O its centre, MN the directing line, and VZ the director of the eye. Draw OD perpendicular to MN, take OE a third proportional to OD, and the radius of the circle; and DL a mean proportional between OD and ED. Bisect the line LV in U, and draw UY perpendicular to VL, meeting MN in Y; with the radius YL, or YV, and centre Y, describe a circle intersecting MN in I and K. Draw the lines IWET and KHEC; then it is evident from the second lemma that, the image of the point E, is the centre C, and that the images of the chords TW and GH, are the axes PQ and RS, of the ellipse, which is the representation of the given circle on the plane of the picture.

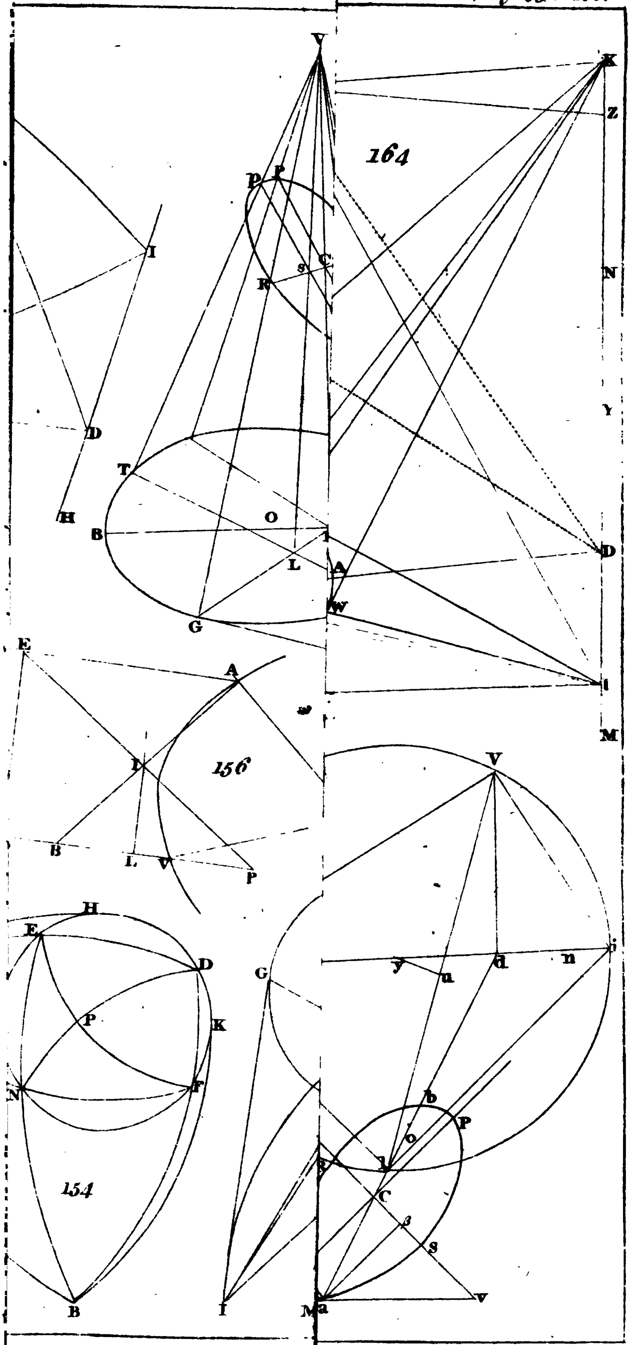
Let *mn* (fig. 166, pl. 8.) be the vanishing line of the plane of the circle, *d* the centre of that line, and V*d* its distance. Also let *ab* be the representation of the diameter AB, which is perpendicular to the directing line, (its vanishing point being *d*,) and *o* the representation of the centre of the circle. Take *dl* a mean proportional between *do* and *d*C, and join V*l*. Bisect V*l* in *u*, and draw *uy* perpendicular to V*l*, meeting *mn* in *y*; with the radius *yl*, or *y*V, describe a circle intersecting *mn* in *i* and *k*, and join *i*, *l*; and *l*, *k*. Through the centre C, draw γCP and vCR parallel to *il* and *kl* respectively, and they will represent the indefinite axes sought. This will appear evident when the corollaries to the third lemma are attentively considered.

To find the lengths of the axes.

Through *a*, the extremity of the diameter *ba*, draw $\gamma a v$ parallel to the vanishing line *mn*, meeting the axes in γ and v, and draw $a\alpha$ and $a\beta$ perpendicular to $C\gamma$ and Cv. Take CQ a mean proportional between γC and αC, and CS a mean proportional between vC and βC; then CQ and CS are the semi-axes sought.

For since a tangent to the circle at the point A is parallel to *mn*, its image will also be parallel to *mn*, and a tangent to the ellipse at *a*; therefore $\gamma a v$ is a tangent to the curve at *a*; and by a well known property of the ellipse, the squares of the semi-diameters CQ and CS, are equal to the rectangles $\gamma C \times C\alpha$ and $vC \times C\beta$, which they are by construction, therefore CQ and CS are the semi-axes required.

It is obvious that the image of the given circle will be an ellipse, a parabola, or an hyperbola, according as the circle falls without, touches, or is cut by the directing line. It is the first of these cases only that we have considered in the above solution. A complete solution, however, of all the cases may be found in Hamilton's Perspective, B. III.



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XXVIII. QUESTION 238, *Mr.* IVORY.

Let n be any whole poſitive number, and put $Q = \frac{2.4.6.8......2n}{3.5.7.9......(2n+1)}$: then the whole value of the triple fluent $\int\frac{dx}{\sqrt{(1-x^2)}}\int\frac{dx}{\sqrt{(1-x^2)}}\int\frac{x^{2n+1}dx}{\sqrt{(1-x^2)}}$, which is generated while x increaſes from 0 to 1, is equal to

$$Q\times\left\{\frac{1}{(2n+3)^2}+\frac{1}{(2n+5)^2}+\frac{1}{(2n+7)^2}+\frac{1}{(2n+9)^2}+\&c.\right\}?$$

N.B. Obſerve that the caſe $n=0$ is included by taking $Q = 1$.

FIRST SOLUTION, *by Mr.* LOWRY.

It is well known that

$$\int\frac{x\dot{x}}{\sqrt{(1-x^2)}} = 1-\sqrt{(1-x^2)}$$

$$\int\frac{x^3\dot{x}}{\sqrt{(1-x^2)}} = \frac{2}{3}-\frac{2}{3}(1+\tfrac{1}{2}x^2)\sqrt{(1-x^2)}$$

$$\int\frac{x^5\dot{x}}{\sqrt{(1-x^2)}} = \frac{2.4}{3.5}-\frac{2.4}{3.5}(1+\tfrac{1}{2}x^2+\frac{1.3}{2.4}x^4)\sqrt{(1-x^2)}$$

&c. &c.

and in general, that $\int\frac{x^{2n+1}\dot{x}}{\sqrt{(1-x^2)}} =$

$$Q-Q(1+\tfrac{1}{2}x^2+\frac{1.3}{2.4}x^4+\frac{1.3.5}{2.4.6}x^6+..\frac{1.3.5..2n-1}{2.4.6...2n}x^{2n})\sqrt{(1-x^2)};$$

where the first term of each expression is the fluent generated while x increases from 0 to 1.

Therefore the double fluent $\int\frac{\dot{x}}{\sqrt{(1-x^2)}}\int\frac{x^{2n+1}\dot{x}}{\sqrt{(1-x^2)}}$ is $=$

$$Q\int(\frac{\dot{x}}{\sqrt{(1-x^2)}}-\dot{x}-\tfrac{1}{2}x^2\dot{x}-\frac{1.3}{2.4}x^4\dot{x}-...\frac{1.3.5..2n-1}{2.4.6...2n}x^{2n}\dot{x})$$

$$= Q\,(A-x-\frac{1}{2.3}x^3-\frac{1.3}{2.4.5}x^5-..\frac{1.3.5..2n-1}{2.4.6..2n+1}x^{2n+1}),$$

A being the circular arc whose sine is x. And if R be put for the triple fluent $\int\frac{\dot{x}}{\sqrt{(1-x^2)}}\int\frac{\dot{x}}{\sqrt{(1-x^2)}}\int\frac{x^{2n+1}\dot{x}}{\sqrt{(1-x^2)}}$ we have

$$R = Q\int\frac{\dot{x}}{\sqrt{(1-x^2)}}\left(A - x - \frac{1}{2\cdot3}x^3 - \frac{1\cdot3}{2\cdot4\cdot5}x^5 - \ldots\frac{1\cdot3\cdot5\ldots2n-1}{2\cdot4\cdot6\ldots2n+1}x^{2n+1}\right).$$

But $\frac{\dot{x}}{\sqrt{(1-x^2)}}$ being $=\dot{A}$, the fluent of the first term $\frac{A\dot{x}}{\sqrt{(1-x^2)}}$ is $=\frac{A^2}{2}$; and the fluents $\int\frac{x\dot{x}}{\sqrt{(1-x^2)}}$, $\int\frac{x^3\dot{x}}{\sqrt{(1-x^2)}}$, $\int\frac{x^5\dot{x}}{\sqrt{(1-x^2)}}$, to $\frac{x^{2n+1}\dot{x}}{\sqrt{(1-x^2)}}$, generated while x increases from 0 to 1, being $1, \frac{2}{3}, \frac{2\cdot4}{3\cdot5}, \frac{2\cdot4\cdot6}{3\cdot5\cdot7}$, to $\frac{2\cdot4\cdot6\ldots\ldots2n}{3\cdot5\cdot7\ldots\ldots2n+1}$ respectively; we have, by substitution,

$$R = Q\left(\frac{A^2}{2} - 1 - \frac{1}{3^2} - \frac{1}{5^2} - \frac{1}{7^2} - \ldots\ldots\frac{1}{(2n+1)^2}\right).$$

But when $x = 1$, A is a quadrant, and $\frac{A^2}{2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} +$ &c. continued indefinitely (see the solution to qu. 76); therefore $\frac{A^2}{2} - 1 - \frac{1}{3^2} - \frac{1}{5^2} - \frac{1}{7^2} - \ldots\frac{1}{(2n+1)^2}$, must be equal to the remaining terms $\frac{1}{(2n+3)^2}$, $\frac{1}{(2n+5)^2}$, $\frac{1}{(2n+7)^2}$, &c. and $\int\frac{\dot{x}}{\sqrt{(1-x^2)}}\int\frac{\dot{x}}{\sqrt{(1-x^2)}}\int\frac{x^{2n+1}\dot{x}}{\sqrt{(1-x^2)}} =$

$$Q\left(\frac{1}{(2n+3)^2} + \frac{1}{(2n+5)^2} + \frac{1}{(2n+7)^2} + \frac{1}{(2n+9)^2} + \&c.\right).$$

SECOND SOLUTION, *by Mr.* CUNLIFFE.

Assume $\int\frac{\dot{x}}{\sqrt{(1-x^2)}}\int\frac{x^{2n+1}\dot{x}}{\sqrt{(1-x^2)}} = ax^{2n+3} + bx^{2n+5} + cx^{2n+7} +$ &c.

Taking the fluxions and dividing by $\frac{\dot{x}}{\sqrt{(1-x^2)}}$ gives $\int\frac{x^{2n+1}\dot{x}}{\sqrt{(1-x^2)}}$ $=$

$$=\sqrt{(1-x^2)}\times\left\{(2n+3)ax^{2n+2}+(2n+5)bx^{2n+4}+(2n+7)cx^{2n+6}+\&c.\right\}$$

Again, taking the fluxions and dividing by $\frac{x^{2n+1}\dot{x}}{\sqrt{(1-x^2)}}$ we have

$$1=(1-x^2).\left\{(2n+2)(2n+3)a+(2n+4)(2n+5)bx^2+(2n+6)(2n+7)cx^4+\&c.\right\}$$
$$-(2n+3)ax^2-(2n+5)bx^4-(2n+7)cx^6-(2n+9)dx^8-\&c.$$

By actual multiplication, &c.

$$\left.\begin{array}{r}(2n+2)(2n+3)a+(2n+4)(2n+5)bx^2+(2n+6)(2n+7)cx^4\ \&c.\\ -1-(2n+2)(2n+3)ax^2-(2n+4)(2n+5)bx^4\ \&c.\\ -(2n+3)ax^2-\qquad\qquad(2n+5)bx^4\ \&c.\end{array}\right\}=0.$$

Making the homologous terms $=0$, $a=\frac{1}{(2n+2)(2n+3)}$,

$$b=\frac{(2n+3)}{(2n+2)(2n+4)(2n+5)},\ c=\frac{(2n+3)(2n+5)}{(2n+2)(2n+4)(2n+6)(2n+7)},\ \&c.$$

$$\text{Again,}\int\frac{\dot{x}}{\sqrt{(1-x^2)}}\int\frac{\dot{x}}{\sqrt{(1-x^2)}}\int\frac{x^{2n+1}\dot{x}}{\sqrt{(1-x^2)}}=\int\frac{\dot{x}}{\sqrt{(1-x^2)}}\times\left\{ax^{2n+3}+bx^{2n+5}\right.$$
$$\left.+cx^{2n+7}+\&c.\right\}=\int\frac{x^{2n+1}\dot{x}}{\sqrt{(1-x^2)}}\times\left\{ax^2+bx^4+cx^6+dx^8\ \&c.\right\}$$

the fluent of which generated whilst x from 0 becomes 1, is Q

$$\times\left\{\frac{(2n+2)a}{(2n+3)}+\frac{(2n+2)(2n+4)b}{(2n+3)(2n+5)}+\frac{(2n+2)(2n+4)(2n+6)c}{(2n+3)(2n+5)(2n+7)}+\&c.\right.$$

where Q denotes the whole fluent of $\frac{x^{2n+1}\dot{x}}{\sqrt{(1-x^2)}}$. (See art. 286, vol. 2. Simpson's Fluxions.)

Now $Q=\frac{2.4.6\ldots.(n)}{1.3.5.7..(n)}=\frac{2.4.6\ldots.(2n)}{3.5.7\ldots.(2n+1)}$, by art. 298 of the same book. And by writing for a, b, c, &c. their values before determined, we get

$$Q\times\left\{\frac{(2n+2)a}{(2n+3)}+\frac{(2n+2)(2n+4)b}{(2n+3)(2n+5)}+\frac{(2n+2)(2n+4)(9n+6)c}{(2n+3)(2n+5)(2n+7)}+\&c.\right\}$$
$$=Q\times\left\{\frac{1}{(2n+3)^2}+\frac{1}{(2n+5)^2}+\frac{1}{(2n+7)^2}+\frac{1}{(2n+9)^2}+\&c.\right\}.$$

XXIX. QUESTION 239, *by Mr.* IVORY.

Let n be a whole poſitive number, and put $P=\frac{1.3.5.7\ldots.(2n-1)}{2.4.6.8\ldots.2n}$: then the whole value of the triple fluent $\int\frac{dx}{\sqrt{(1-x^2)}}\int\frac{dx}{\sqrt{(1-x^2)}}\int\frac{x^{2n}dx}{\sqrt{(1-x^2)}}$, which is generated while x increaſes from o to 1, is equal to

$$P\times\left\{\frac{1}{(2n+2)^2}+\frac{1}{(2n+4)^2}+\frac{1}{(2n+6)^2}+\frac{1}{(2n+8)^2}+\&c.\right\}\times\frac{\pi}{2}.$$

where π is the ſemi-circumference to the radius 1 ?

N. B. Obſerve that the caſe $n = 0$ is included by taking $P = 1$.

FIRST SOLUTION, *by Mr.* LOWRY.

Let A = the arch whose sine is x; then

$$\int\frac{\dot{x}}{\sqrt{(1-x^2)}}=A,\int\frac{x^2\dot{x}}{\sqrt{(1-x^2)}}=\tfrac{1}{2}A-\tfrac{1}{2}x\sqrt{(1-x^2)}$$

$$\int\frac{x^4\dot{x}}{\sqrt{(1-x^2)}}=\frac{1\cdot3}{2\cdot4}A-\frac{1\cdot3}{2\cdot4}(x+\tfrac{2}{3}x^3)\sqrt{(1-x^2)}$$

&c. &c.

and in general $\int\frac{x^{2n}\dot{x}}{\sqrt{(1-x^2)}}=$

$$PA-P(x+\tfrac{2}{3}x^3+\frac{2\cdot4}{3\cdot5}x^5\ldots\ldots\frac{2\cdot4\ldots\ldots2n-2}{1\cdot3\cdot5\,..\,2n-1}x^{2n-1})\sqrt{(1-x^2)};$$

where the first term of each expression, when $A=\frac{\pi}{2}$, is the whole fluent generated while x increases from o to 1.

Therefore $\int\frac{\dot{x}}{\sqrt{(1-x^2)}}\int\frac{x^{2n}\dot{x}}{\sqrt{(1-x^2)}}=$

$$P\int\left\{A\dot{A}-(x\dot{x}+\tfrac{2}{3}x^3\dot{x}\ldots\ldots\frac{2\cdot4\ldots\ldots2n-2}{1\cdot3\cdot5\,..\,2n-1}x^{2n-1}\dot{x})\right\}$$

$$=P(\frac{A^2}{2}-\frac{x^2}{2}-\frac{2}{3\cdot4}x^4-\ldots\ldots\frac{2\cdot4\ldots\ldots2n-2}{3\cdot5\cdot7\ldots\ldots2n}x^{2n}).$$

And if R be put for the triple fluent

$\int\frac{x}{\sqrt{(1-x^2)}}\int\frac{\dot{x}}{\sqrt{(1-x^2)}}\int\frac{x^{2n}\dot{x}}{\sqrt{(1-x^2)}}$, we have R =

$$P\int\frac{A^2\dot{A}}{2}+P\int\frac{\dot{x}}{\sqrt{(1-x^2)}}(-\frac{x^2}{2}-\frac{2}{3\cdot4}x^4\ldots\ldots\frac{2\cdot4\ldots\ldots2n-2}{3\cdot5\cdot7\ldots\ldots2n}x^{2n}).$$

But the fluents $\int\frac{A^2\dot{A}}{2}$, $\int\frac{x^2\dot{x}}{\sqrt{(1-x^2)}}$, $\int\frac{x^4\dot{x}}{\sqrt{(1-x^2)}}$ to

$\int \frac{x^{2n}\dot{x}}{\sqrt{(1-x^2)}}$, generated while x increases from 0 to 1, being $\frac{\pi^2}{24}\cdot\frac{\pi}{2}$; $\frac{1.\pi}{2.2}$, $\frac{1.3.\pi}{2.4.2}$, $\frac{1.3.5.\pi}{2.4.6.2}$ to $\frac{1.3.5\ldots.2n-1}{2.4.6\ldots.2n}\cdot\frac{\pi}{2}$, respectively; we have by substitution,

$$R = P\left(\frac{\pi^2}{24} - \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{6^2} - \frac{1}{8^2} - \ldots.\frac{1}{(2n)^2}\right)\cdot\frac{\pi}{2}.$$

But $\frac{\pi^2}{24} = \frac{\pi^2}{6} - \frac{\pi^2}{8} = (1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.) - (1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c.) = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \&c.$ continued indefinitely (see solution to qu. 179).

Therefore $\frac{\pi^2}{24} - \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{6^2} - \frac{1}{8^2} - \ldots.\frac{1}{(2n)^2}$ is equal to the remaining terms $\frac{1}{(2n+2)^2}$, $\frac{1}{(2n+4)^2}$, $\frac{1}{(2n+6)^2} + \&c.$

And $\int\frac{\dot{x}}{\sqrt{(1-x^2)}}\int\frac{\dot{x}}{\sqrt{(1-x^2)}}\int\frac{x^{2n}\dot{x}}{\sqrt{(1-x^2)}} =$

$$P\left(\frac{1}{(2n+2)^2} + \frac{1}{(2n+4)^2} + \frac{1}{(2n+6)^2} + \&c.\right)\times\frac{\pi}{2}.$$

SECOND SOLUTION, *by Mr.* CUNLIFFE.

By assuming $\int\frac{\dot{x}}{\sqrt{(1-x^2)}}\int\frac{x^{2n}\dot{x}}{\sqrt{(1-x^2)}} = ax^{2n+2} + bx^{2n+4} + cx^{2n+6}$ + &c. and proceeding as in qu. 238, we shall get $a = \frac{1}{(2n+1)(2n+2)}$, $b = \frac{(2n+2)}{(2n+1)(2n+3)(2n+4)}$, $c = \frac{(2n+2)(2n+4)}{(2n+1)(2n+3)(2n+5)(2n+6)}$, &c. and therefore

$$\int\frac{\dot{x}}{\sqrt{(1-x^2)}}\int\frac{\dot{x}}{\sqrt{(1-x^2)}}\int\frac{x^{2n}\dot{x}}{\sqrt{(1-x^2)}} = \int\frac{\dot{x}}{\sqrt{(1-x^2)}}\times\left\{ax^{2n+2} + bx^{2n+4} + cx^{2n+6} + \&c.\right\} = \int\frac{x^{2n}\dot{x}}{\sqrt{(1-x^2)}}\times\left\{ax^2 + bx^4 + cx^6 + \&c.\right\}.$$

Now by art. 286, vol. 2, Simpson's Fluxions, the fluent of the last expression generated while x from 0 becomes 1, is Pq ×

$$\left\{\frac{(2n+1)a}{(2n+2)}+\frac{(2n+1)(2n+3)b}{(2n+2)(2n+4)}+\frac{(2n+1)(2n+3)(2n+5)c}{(2n+2)(2n+4)(2n+6)}+\&c.\right\}$$

where Pq denotes the whole fluent of $\frac{x^{2n}\dot{x}}{\surd(1-x^2)}$. And by art. 298, of the same book, $Pq=q\times\frac{1.3.5.7(n)}{2.4.6.8(n)}=q\times\frac{1.3.5..(2n-1)}{2.4.6..(2n)}$, where q denotes the quadrantal arc of a circle, radius 1.

Again, by writing the above values of a, b, c, &c. Pq

$$\times\left\{\frac{(2n+1)a}{(2n+2)}+\frac{(2n+1)(2n+3)b}{(2n+2)(2n+4)}+\frac{(2n+1)(2n+3)(2n+5)c}{(2n+2)(2n+4)(2n+6)}+\&c.\right\}$$
$$=\frac{Pq}{4}\times\left\{\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\frac{1}{(n+3)^2}+\frac{1}{(n+4)^2}+\&c.\right\}.$$

XXX. PRIZE QUESTION 240, *by Mr.* IVORY.

Let a plane be drawn through the axis of an oblique cone, so as to be perpendicular to the plane of the base; and let a right-line be drawn to bisect the vertical angle of the triangle which is the common section of the plane and the cone: then, if a second plane be drawn through this line (*viz.* the line which bisects the vertical angle of the triangle) to cut the cone, the difference of two portions of the conic surface between the two planes, and comprehended in opposite angles, may be determined by means of a circular area and a rectilineal space?

SOLUTION, *by Mr.* IVORY, *the Proposer.*

Let CP (fig. 30. pl. D.) the height of the cone $=h$: the distance of the foot of CP from the centre of the base, OP, $=k$: the radius of the base $=r$: the distance of the point E, through which the plane CDF passes, $=r\times f$: the measure of the indefinite arc AD, $=\varphi$: and the line CD, in the conic surface, $=\varsigma$. Then, $CD^2 = CP^2 + PR^2 + RD^2$; that is, in symbols,

$$\varsigma^2 = h^2 + (k + r\cos\varphi)^2 + r^2\sin^2\varphi,$$
$$\text{or } \varsigma^2 = h^2 + k^2 + r^2 + 2rk\cos\varphi:$$

The angle AED being denoted by ψ, then

$$\sin(\varphi - \psi) = f \sin \psi$$
$$\cos(\varphi - \psi) = \sqrt{(1 - f^2 \sin^2 \psi)}$$
$$\text{and } \cos \varphi = \cos(\varphi - \psi) \cos \psi - \sin(\varphi - \psi) \sin \psi$$
$$= \cos \psi \sqrt{(1 - f^2 \sin^2 \psi)} - f \sin^2 \psi.$$

Again, let AC $= a$, and BC $= b$, then

$$a^2 = h^2 + (k + r)^2$$
$$b^2 = h^2 + (k - r)^2$$

consequently $\frac{a^2 + b^2}{2} = h^2 + k^2 + r^2 = \left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2$

$$\frac{a^2 - b^2}{2} = 2rk = 2\left(\frac{a+b}{2}\right) . \left(\frac{a-b}{2}\right).$$

Therefore by substitution,

$$\varrho^2 = \left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2 - 2\left(\frac{a+b}{2}\right)\left(\frac{a-b}{2}\right) f \sin^2 \psi$$
$$+ 2\left(\frac{a+b}{2}\right)\left(\frac{a-b}{2}\right) \cos \psi \sqrt{(1 - f^2 \sin^2 \psi)}$$

$$\text{or, } \varrho^2 = \left(\frac{a+b}{2}\right)^2 \times \left\{ \begin{array}{l} 1 + \left(\frac{a-b}{a+b}\right)^2 - 2\left(\frac{a-b}{a+b}\right) f \sin^2 \psi \\ + 2\left(\frac{a-b}{a+b}\right) \cos \psi \sqrt{(1 - f^2 \sin^2 \psi)} \end{array} \right\}$$

but, according to the enunciation of the proposition, the point E is taken so that $f = \frac{a-b}{a+b}$: therefore

$$\varrho^2 = \left(\frac{a+b}{2}\right)^2 \times \left\{ 1 + f^2 - 2f^2 \sin^2 \psi + 2f \cos \psi \sqrt{(1 - f^2 \sin^2 \psi)} \right\}$$

and, by extracting the square root of both sides,

$$\varrho = \frac{a+b}{2} \times \left\{ f \cos \psi + \sqrt{(1 - f^2 \sin^2 \psi)} \right\}.$$

Because $\sin(\varphi - \psi) = f \sin \psi$
$\cos(\varphi - \psi) = \sqrt{(1 - f^2 \sin^2 \psi)}$

we have $d\varphi - d\psi = \frac{d.\sin(\varphi - \psi)}{\cos(\varphi - \psi)} = \frac{f \cos \psi \,.\, d\psi}{\sqrt{(1 - f^2 \sin^2 \psi)}}$:

therefore $d\varphi = \frac{d\psi}{\sqrt{(1 - f^2 \sin^2 \psi)}} \times \left\{ f \cos \psi + \sqrt{(1 - f^2 \sin^2 \psi)} \right\}$:

also (because $\frac{a-b}{2} = \frac{a+b}{2} \times f$)

$$d\varrho = \frac{a-b}{2} \times \frac{-d\psi \sin \psi}{\sqrt{(1 - f^2 \sin^2 \psi)}} \times \left\{ f \cos \psi + \sqrt{(1 - f^2 \sin^2 \psi)} \right\},$$

therefore by substitution, $\varrho \sqrt{(r^2 d\varphi^2 - d\varrho^2)} =$

$$\frac{(a+b)r}{2} \times \frac{d\psi \sqrt{(1 - (\frac{a-b}{2r})^2 \sin^2 \psi)}}{\sqrt{(1 - f^2 \sin^2 \psi)}} \times \left\{ f \cos\psi + \sqrt{(1 - f^2 \sin^2 \psi)} \right\}^2.$$

Now $\varrho \sqrt{(r^2 d\varphi^2 - d\varrho^2)}$ is double of the fluxion of the part of the conic surface bounded by the lines AC and CD*, which space is therefore (by putting $e = \frac{a-b}{2r}$) =

$$(\frac{a+b}{4})r \times \int \frac{d\psi \sqrt{(1-e^2 \sin^2\psi)}}{\sqrt{(1-f^2 \sin^2\psi)}} \times \left\{ f\cos\psi + \sqrt{(1-f^2\sin^2\psi)} \right\}^2.$$

In the very same manner it may be shewn that the part of the conic surface bounded by the lines CB and CF is equal to

$$(\frac{a+b}{4})r \times \int \frac{d\psi \sqrt{(1-e^2 \sin^2\psi)}}{\sqrt{(1-f^2 \sin^2\psi)}} \times \left\{ -f\cos\psi + \sqrt{(1-f^2\sin^2\psi)} \right\}^2;$$

Therefore the difference of the two said spaces is equal

$$(a+b)fr \times \int d\psi \cos\psi \sqrt{(1 - e^2 \sin^2 \psi)}$$

or to $(a+b)r \times \frac{f}{e} \times \int e d\psi \cos\psi \sqrt{(1 - e^2 \sin^2 \psi)}$

or, by making $x = e \sin\psi$, to $2r^2 \times \int dx \sqrt{(1 - x^2)}$.

Therefore by integrating, the difference of the two parts of the conic surface, that stand upon the arcs AD and BF, is equal to

$$r^2 \times \left\{ x \sqrt{(1 - x^2)} + \text{Arc} (\sin = x) \right\}.$$

Where $x = \frac{a-b}{2r} \times \sin\psi$.

This formula suggests the following construction: Describe the circle *adbe* (fig. 31. pl. D.), equal to the base of the cone; in a diameter *ab*, take *of*, from the centre *o*, $= \frac{a-b}{2}$; draw *ed* through *f* so as to make the angle $afd = \psi$; and, lastly, draw the diameter *el*: then will the mixtilineal space *led* be equal to the difference of the two conic spaces between the planes (fig. 30. pl. D.) AFBC and DFEC, and comprehended in opposite angles.

The difference of the parts of the conic surface that lie on the opposite sides of the plane CFD (fig. 30.) is double of the space *led* (fig. 31.). For the difference of the conic spaces that stand on the arcs AF and BD is equal to the difference of the spaces that stand on the arcs AD and BF.

Mr. Ivory is requested to send to Mr. Glendinning's for the Medal for solving the Prize Question.

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* See the second solution to question 16, page 39, vol. I.

NOTICES.

I. MATHEMATICAL WORKS LATELY PUBLISHED.

The Philosophical Transactions of the Royal Society of London, for 1808. The Mathematical Papers contained in this vol. are,

1. On a New Property of the Tangents of the three angles of a Plane Triangle. By Mr. Wm. Garrad, Quarter-Master of Instruction at the Royal Naval Asylum, at Greenwich.

2. On a New Property of the Tangents of three Arches trisecting the Circumference of a Circle. By Nevil Maskeline, D.D. F. R. S. and Astronomer Royal.

3. Observations of a Comet, made with a view to investigate its Magnitude and the nature of its Illumination. To which is added, an Account of a New Irregularity lately perceived in the apparent figure of the Planet Saturn. By Dr. Herschell.

4. Hydraulic Investigations, subservient to an intended Croonian Lecture on the Motion of the Blood. By Dr. Young.

The Transactions of the Royal Society of Edinburgh. Vol. VI. Part II. contains the following Mathematical Papers.

1. Of the Solids of Greatest Attraction, or those which, among all the Solids that have certain Properties, attract with the greatest Force in a given Direction. By Professor Playfair.

2. An Account of a very extraordinary Effect of Refraction, observed at Ramsgate, by the Rev. S. Vince.

3. New Series for the Quadrature of the Conic Sections, and the Computation of Logarithms. By Mr. Wm. Wallace.

An Essay on the various orders of Logarithmic Transcendentals. With an inquiry into their application to the Integral Calculus and the Summation of Series. By Wm. Spence. Quarto. p. p. 128. Price 12s.

II. MATHEMATICAL WORKS IN THE PRESS.

A New and Enlarged edition of Mr. Professor Playfair's Proofs of the Huttonian Theory of the Earth is in the Press, and will speedily be published in one volume, quarto.

Mr. Renoward, of Trinity College, Cambridge, has in the Press a Treatise on Spherical Trigonometry.

A System of Conic Sections, adapted to the Study of Natural Philosophy. By the Rev. D. M. Peacock, A. M. Formerly Fellow of Trinity College, Cambridge.

Tracts, Mathematical and Philosophical, consisting chiefly of original or unpublished Pieces. By Dr. Hutton. These Tracts will form about 2 or 3 Octavo Volumes, and may be considered as Dr. Hutton's select remaining Papers of Inventions, Discoveries, and Improvements in those Sciences.

OBITUARY.

III. Died at Fotheringham, near Dundee, on the 23rd of August, 1808, in the 75th year of his age, Robert Small, D.D, F. R. S. EDIN. One of the Ministers of Dundee. Dr. Small has been long known to the World not only as a learned Divine, but also as an ingenious Mathematician, and he has sufficiently shewn how well he was entitled to this last Character by the excellent Demonstrations he has given of some of Dr. Stewart's General Theorems in the 2nd vol. of the Edin. Phil. Transactions; and more recently by his Account of the Astronomical Discoveries of Kepler, published in the year 1804, in one vol, 8vo,

End of the First Part of the Second Volume,

W. Glendinning, Printer, Hatton Garden, London,

NEW SERIES
OF THE
MATHEMATICAL
REPOSITORY.

VOL. II. PART II.

ORIGINAL ESSAYS.

ARTICLE I.

DEMONSTRATION *of a Propoſition in Mechanics.* By A. B.

To the Editor of the Mathematical Repository.

SIR,

If you can inſert the following Mechanical Propoſition, in your valuable Miſcellany, without excluding matter of more conſequence, you will oblige

Sir,

Your's, &c.

A. B.

9*th. May*, 1806.

PROPOSITION.

If a ſyſtem of bodies be connected together and ſupported at any point which is not the centre of gravity, and then left to deſcend by that part of their weight which is not ſupported; $4l$ multiplied into the ſum of all the products of each body into the ſpace it has perpendicularly deſcended, will be equal to the ſum of all the products of each body into the ſquare of its velocity; l being equal to $16\frac{1}{12}$ feet, or the ſpace through which a heavy body deſcends in one ſecond of time.

Let A, B, C, &c. (fig. 1. pl. A.) be a ſyſtem of bodies connected together and ſupported at S; and let G, O, be the centres of gravity and oſcillation; join SA, SB, SC, &c. Let SD be parallel and SL perpendicular to the horizon, and at the diſtance SO deſcribe the quadrant DQL, and put s and c for the ſine and coſine of the angle OSL to the rad. (1). In any *ſmall time t'*, the velocity generated in a body at O, deſcending down the arch DL, $\pm\ 2t'ls$, and by the property of the centre of oſcillation, the velocity generated in the ſame time in O, is likewiſe $= 2t'l \times s$, and therefore the velocity generated in the centre of gravity G, will $= 2t'ls \times \frac{SG}{SO}$; but by mechanics $SO = \frac{A \times SA^2 + B \times SB^2 + C \times SC^2 + \&c.}{(A + B + C + \&c.) \times SG}$; therefore by ſubſtitution, the velocity generated in the centre of gravity in the time t', $= 2t'ls$

$$\times \frac{SG^2 \times (A + B + C + \&c.)}{A \times SA^2 + B \times SB^2 + C \times SC^2 + \&c.}$$

But becauſe the velocities of A, B, C, &c. and G, are in the ſame proportion as their reſpective diſtances from S, and theſe diſtances are of the ſame dimenſion in both numerator and denominator; if, therefore, for the velocities of A, B, C, &c. and G, we write v, u, w, &c. and V, the equation becomes $2t'ls \times \frac{V^2 \times (A + B + C + \&c.)}{Av^2 + Bu^2 + Cw^2 + \&c.} = V$, or, $2t'ls \times V \times (A + B + C + \&c.) = Av^2 + Bu^2 + Cw^2 + \&c.$ Let v', u', w', &c. and V', be the reſpective velocities of A, B, C, &c. and G, in a direction perpendicular to the horizon, then becauſe $V : V' :: 1 : s$, $V = \frac{V'}{s}$, hence, $2t'lV' \times (A + B + C + \&c.)$

$= Av^2 + Bu^2 + Cw^2 + \&c. = 2t'l \times (-Av' + Bu' + Cw' + \&c.)$ by the nature of the centre of gravity. Suppoſe x, y, z, &c. to repreſent the ſpaces paſſed over in a perpendicular direction in the time t'; then becauſe in equably accelerated motions, the ſpaces paſſed over are as the times multiplied into the laſt velocities, therefore $1'' \times 2l : t' \times v' :: l : x$, therefore $v' = \frac{2x}{t'}$, and in like manner $u' = \frac{2y}{t'}$, $w' = \frac{2z}{t'}$, &c. hence by ſubſtitution $4l \times (-Ax + By + Cz + \&c) = Av^2 + Bu^2 + Cw^2 + \&c.$ Now as neither t' nor s enter into the expreſſion, the reſult is not confined to any particular time or poſition; it is therefore evident that the equation is general as expreſſed in the propoſition; for it is well known to mathematicians, that if in the concluſion of a problem all thoſe quantities vaniſh that reſtricted it to any particular caſe, then the reſult, or equation becomes *general* for thoſe quantities which remain; wherefore the true relation between the ſpaces perpendicularly deſcended by the bodies and the ſquares of their velocities has been determined in all caſes whatever.

Cor. Becauſe the velocity generated in the centre of gravity in a direction perpendicular to SO in the time $t' = 2t'ls \times \frac{SG}{SO}$, the velocity generated in the ſame time in a direction perpendicular to the horizon muſt be equal $2t'ls^2 \times \frac{SG}{SO}$, and $2t'l =$ velocity that would be generated in the ſame time by the whole weight of the ſyſtem (W); therefore $2t'l : 2t'ls^2 \times \frac{SG}{SO} :: W : s^2W \times \frac{SG}{SO} =$ that part of the weight which is not ſupported, or which generates the perpendicular velocity in the centre of gravity. Hence it is evident, that the preſſure upon S at that inſtant is equal to the remaining part of the whole body $= W \times \left(1 - s^2 \times \frac{SG}{SO}\right)$.

Ex. 1. From Simpſon's Tracts, page 131.

Suppoſe that a thread ACnCA (fig. 2. pl. A.), having two equal weights A, A, ſuſpended at the ends thereof, is hung over two tacks C, C, in the ſame horizontal line; and that to the middle point of the thread (n) equally diſtant from the tacks, another given weight B is fixed, which is permitted to deſcend by its own gravity, ſo as to cauſe the other two weights to aſcend : it is propoſed to find the law of the velocity by which the ſaid weights aſcend and deſcend, &c. &c.

Let bn be an indefinitely ſmall ſpace through which the body has deſcended, join Cb and draw br perpendicular to Cn ; then the velocity of A : velocity of B :: rn : bn :: En : Cn by ſimilar triangles. If therefore CE $= a$, C$n = y$, E$n = x$, and $u =$ velocity of B, then the velocity of A $= \frac{ux}{y}$. The ſpace aſcended by A, A $= y - a$ and the ſpace deſcended by B $= x$; therefore, by ſubſtituting in the general theorem we ſhall have

$$4l \times (2Aa - 2Ay + Bx) = 2A \times \frac{u^2x^2}{y^2} + Bu^2;$$

hence, the equation reduced gives

$$w^2 = 4ly^2 \times \frac{Bx - 2Ay + 2Aa}{By^2 + 2Ax^2}, \text{ and therefore}$$

$$u = 2y \sqrt{\left(l \times \frac{Bx - 2Ay + 2Aa}{By^2 + 2Ax^2}\right)} = \text{velocity of B.}$$

The velocity of A, $A = \frac{ux}{y} = 2x \sqrt{(l \times \frac{Bx - 2Ay + 2Aa}{By^2 + 2Ax^2})}$. Simpſon's expreſſion for the velocities of B and A, when properly reduced, perfectly agree with thoſe given above.

Ex. 2. Let the weights A and D (fig. 3. pl. A) be unequal, and ſuppoſe the weight B to deſcend in the ſtraight line EB, E being the centre of gravity of A and D when placed at C and C, and let the ſeveral lines be expreſſed as is repreſented in the figure ; then by ſubſtitution, in the general theorem given above, we have

$$4l \times Bx - A \times (y - a) - D \times (z - b) = A \times \frac{x^2u^2}{y^2} + D \times \frac{x^2u^2}{z^2}.$$

Reduced, $u^2 = 4ly^2z^2 \times \frac{Bx - A \times (y - a) - D \times (z - b)}{Ax^2z^2 + By^2z^2 + Dx^2y^2}$.

If x be given, y and z are known ; E being the centre of gravity of A and D, A$a =$ Db.

If in the value of u^2 juſt found, we ſuppoſe $a = b$, $z = y$, and A $=$ D, u^2 will be the ſame as in the laſt example.

Ex. 3. Let a cylinder of heavy metal roll down a plane which makes an angle with the horizon whoſe ſine and coſine $= s$ and c, and ſuppoſe the friction juſt ſufficient to keep it from ſliding ; it is required to find the angle which another plane makes with the horizon, ſo that the ſame cylinder moving down it, the ſpaces paſſed over by the ſliding and rolling motions in the ſame time, may be to each other as $n : 1$; the friction varying as the preſſure against the plane.

Let S and C = ſine and coſine of the ſecond plane, w = weight of the cylinder, $l = 16\frac{1}{12}$ feet, t = time from the beginning of the motion, x = ſpace paſſed over by the rolling motion, and v = velocity generated in the circumference of the cylinder round its axis in the ſame time; then it is well known that $\frac{v}{\sqrt{2}}$ = correſponding velocity of the centre of gyration of the cylinder. If the ſame body be placed upon different inclined planes, the forces down theſe planes vary as the ſines of the angles which they make with the horizon and the preſsure against them $\propto$ the coſines of the ſame angles. Hence, sw = force down the firſt plane, and the force of friction $= \frac{ws}{3}$ (ſee Emerſon's Mechanics, Prop. 61, Schol.); therefore the friction upon the ſecond plane $= \frac{Cws}{3c}$; but the force down this plane $= Sw$, therefore the remaining force when that of friction is ſubtracted $= Sw - \frac{Cws}{3c} = w \times \frac{3cS - Cs}{3c}$; hence this proportion, as $w \times 1''^2 : w \times \frac{3cS - Cs}{3c} \times t^2 :: l : lt^2 \times \frac{3cS - Cs}{3c}$ = whole ſpace moved down the plane by both the ſliding and rolling motions.

The force $\frac{Cws}{3c}$ is entirely employed in generating the rotatory motion of the cylinder, and may be conſidered as a weight without inertia, deſcending through the ſpace x; therefore by the theorem $4l \times \frac{Cws}{3c} \times x = w \times \frac{v^2}{2} = w \times \frac{2x^2}{t^2}$, for by mechanics $v = \frac{2x}{t}$, hence $lt^2 \times \frac{2Cs}{3c} = x$ = ſpace paſſed over by rolling; therefore ſpace paſſed over by ſliding $= lt^2 \times \left(\frac{2cS - Cs}{3c} - \frac{2Cs}{3c}\right) = lt^2 \times \frac{3cS - 3Cs}{3c}$. Hence ſpace paſſed over by ſliding : ſpace paſſed over by rolling :: $3cS - 3Cs : 2Cs :: n : 1$, by the hypotheſis; therefore $3(cS - Cs) = 2nCs$, or $3cS = (2n + 3)\,Cs$, and $9c^2S^2 =$

(

$(2n+3)^2 \times C^2s^2 = (2n+3)^2 \times (1 - S^2) \times s^2$; reduced $S^2 = \frac{(2n+3)^2 \times s^2}{9c^2+(2n+3)^2 \times s^2}$, therefore $S = \frac{(2n+3)s}{\surd\{9c^2+(2n+3)^2 \times s^2\}}$.

Cor. If t be given, the space passed over by both or either of these motions will be known; and if the length of the plane, or the space passed over be given, the time t may be found.

The same theorem much facilitates the solution of a great number of difficult mechanical problems, and applies with much advantage to a very considerable part of the propoſitions in Sect. 6. Atwood on Rectilinear and Rotatory Motion. For example, take Prop. 13.

Put x = distance of the centre of gyration from the centre of gravity G, s = space deſcended by P in the time t, and v = velocity generated in P, the other ſymbols as in that Prop.

Then $d : x :: v : \frac{vx}{d}$ = velocity of the centre of gyration; hence, by the theorem, $4l \times Ps = P \times v^2 + w \times \frac{v^2x^2}{d^2} = \frac{Pd^2 + wx^2}{d^2} \times v$.

But by mechanics $v = \frac{2s}{t}$, therefore by ſubſtitution and reduction, it will appear that $x^2 = \frac{lt^2d^2P - sd^2P}{ws} = \frac{Pd^2}{w} \times \frac{lt^2 - s}{s}$, therefore $x = d\sqrt{\left(\frac{P}{w} \times \frac{lt^2 - s}{s}\right)}$ = diſtance of the centre of gyration required. Becauſe $4lP \times \frac{tv}{2} = \frac{Pd^2 + wx^2}{d^2} \times v^2$, therefore $v = 2l \times \frac{Ptd^2}{Pd^2+wx^2} = 2l \times \frac{Pd^2}{Pd^2+wx^2}$ when $t = 1$, and is therefore = the accelerating force when the force of gravity = $2l$. But $x^2 = bd$, therefore the accelerating force of P, or the point D $= v = 2l \times \frac{Pd^2}{Pd^2 + bdw} = 2l \times \frac{Pd}{Pd + bw}$.

ARTICLE II.

On the Motion of Pendulums whose points of ſuſpenſion are moveable. In a Letter from Mr. John Gough.

To the Editor of the Mathematical Repoſitory.

Middleshaw, near Kendal.
12th. May 1806.

SIR,

The Author of the following Propoſitions, is not ignorant that they contradict a remark made on the anſwer to the 648th queſtion of the Gentleman's Diary, by a late eminent mathematician. The knowledge of this circumſtance does not however forbid him to hope that his eſſay will meet with a candid reception on the part of your readers, ſhould it be thought worthy a place in the Mathematical Repoſitory; becauſe he has endeavoured to oppoſe a careful inveſtigation of the ſubject to bare aſſertion; which is of no value when unſupported by demonſtrative evidence, for the greateſt men are ſubject to error as oft as they neglect rigid demonſtration, which is the only true road to geometrical certainty.—The elementary work quoted in the preſent paper, is a Treatiſe of Mechanics lately publiſhed, by Mr. Olinthus Gregory of the Royal Military Academy at Woolwich, which cannot fail of exciting attention, not only by its copiouſneſs and the quantity of original matter, but alſo by the many valuable and judicious ſelections the author has made from the foreign mathematicians.

I am, &c.

JOHN GOUGH.

PROPOSITION I.

IF a given pendulum vibrate upon a pin, fixed in the centre of gravity of a given veſſel, which is ſupported by a perfectly ſmooth and horizontal plane; the common centre of gravity of the pendulum and the veſſel will deſcend and aſcend alternately, in a given right line perpendicular to the horizon; while the centre of gravity of the pendulum deſcribes a portion of an ellipſe, having for its ſemi-tranſverſe the given vertical line properly produced, and for its ſemi-conjugate the diſtance of the ſame centre of gravity from that of the ſyſtem.

DEMONSTRATION. Let the vertical plane AC (fig. 4. pl. A.) repreſent the veſſel, or more properly, that ſection of it, in which the pendulum PG vibrates, P the centre of gravity of AC; alſo let PG be a poſition of the pendulum, G its centre of gravity, and T the common centre of the ſyſtem AC PG; through P draw FE parallel to the horizon, and through T, NM perpendicular to FE. Now the external forces which give motion to the ſyſtem ACPG are, 1ſt. the weight of the pendulum which acts upon its centre of gravity G (Mechanics book 1, art. 106); 2*d*. the reaction of the plane CD, upon the point P, which is parallel to the former force, but in a contrary direction; conſequently the reſultant of theſe two forces is equal to their difference (ibid. 73); and by the nature of the centre of gravity it acts upon the ſyſtem ACPG at the point T, and in the direction NM. But all the other forces, acting upon T, ariſe from the mutual action and reaction of AC and PG; therefore the latter forces do not diſturb the motion of T, their common centre of gravity; hence it follows, from the doctrine of forces, that T will always be found in the right line NM, the poſition of which is thus found. Let p be the place of the centre of gravity of AC, when PG is in an horizontal poſition, coinciding with FE; make $pN = PT$, which is given in magnitude; therefore the point N is given, and NM is perpendicular to FE. Make $NM = PT$, NK and $pR = PG$; then $NR = MK = TG$; and when P coincides with p, G will be at R; alſo when P has moved through $pN = PT$, G will be at K; hence RGK is a quadrant of an ellipſe, having NK for its ſemi-tranſverſe and NR for its ſemi-conjugate (Em. Con. Ellip. prop. 30). *Q. E. D.*

COR. 1. Let $b =$ the weight of the pendulum, $w' =$ that of the veſſel; $l = PG$, the diſtance between the point of ſuſpenſion and the pendulum's centre of gravity; and we have $\frac{bl}{b + w'} = PT = pN$; hence if $w' = 0$ (fig. 4, pl. A.). $pN = PR$ and G deſcends in a vertical line paſſing through R while P moves through the ſpace pR.

COR. 2. But if w' be infinite, pN or its equal $\frac{bl}{b + w'}, = 0$, in which caſe N coincides with p, $NR = NK$; that is, the ellipſe becomes a circle and the point of ſuſpenſion is fixed at p.

FROPOSITION

PROPOSITION II.

LET PG be a pendulum (fig. 4. pl. A.) whoſe point of ſuſpenſion is P, and let it be urged perpendicular to PG, by a force k acting at G, its centre of gravity; then put b = matter in PG, r = radius of gyration of PG to the centre G, q = its radius of gyration to the centre P; and the reaction of the pin P, perpendicular to the line PG, will be $= \frac{r^2k}{q^2+r^2}$.

DEMONSTRATION. For the pendulum, when in motion, muſt revolve about the points G and P with equal angular velocities, conſequently the force k is divided into two leſs forces, one acting at G and the other at P perpendicular to PG (Mech. art. 74.): put the latter $= x$, and the former $= k - x$; then the angular velocity, generated in a given time and about the centre G by the force x, is in a conſtant proportion to $\frac{GP \times x}{br^2}$ (ibid. art. 301, cor. 2, and art. 310.); for the ſame reaſon, the angular velocity generated about P in the ſame time by $k - x$ has an equal ratio to $\frac{PG \times (k-x)}{bq^2}$: but the two angular velocities are equal; hence $\frac{x}{r^2} = \frac{k-x}{q^2}$; conſequently $x = \frac{r^2k}{q^2+r^2}$. *Q. E. D.*

COR. Draw GS perpendicular to FE; alſo put PG $= l$, PS $= n$, $r^2 + q^2 = c^2$, and that part of b which acts perpendicular to PG $= \frac{nb}{l} = k$; hence $x = \frac{r^2nb}{lc^2}$ therefore $k - x = \frac{q^2nb}{lc^2}$: hence that part of $k - x$, which acts parallel to SP, in the direction NT $= \frac{q^2n^2b}{l^2c^2} = f$, or the motive force of the ſyſtem ACPG. Conſequently if there be two pendulums PG, QI, which are every way equal, and make equal angles with the horizontal lines FE, *fe* (fig. 4, pl. A.), the motive forces acting upon T, *t*, the centres of gravity of the ſyſtems ACPG, *ac*QI, will be equal; becauſe the quantities b, l, n, q, r, are equal in both caſes.

PROPOSITION III.

If two bodies A, C, (fig. 5. pl. A.) be impelled in the right lines AB, CD, by motive forces, which are conſtantly equal to each other, though each force may be variable in reſpect of itſelf; the ſpaces deſcribed by A, C, in equal times, will always be inverſely as their quantities of matter or weights, that is, as $\frac{1}{A}$ to $\frac{1}{C}$.

Demonstration. Put y, z = theſe ſpaces; t, v, the correſponding velocities of A, C; then ſince the motive force of A is always equal to that of C, the motion generated in A will always be equal to that generated in C in the ſame time. But the velocity is inverſely as the matter when the motion is given: hence $t : v :: \frac{1}{A} : \frac{1}{C}$; but as $t : v :: \dot{y} : \dot{z}$; therefore $\dot{y} : \dot{z} :: \frac{1}{A} : \frac{1}{C}$; conſequently y is to z as $\frac{1}{A}$ is to $\frac{1}{C}$ (Emerſon's Fluxions, prop. 2, ſect. 1.) *Q. E. D.*

Cor. If two ſpaces, AB, CD, run over by two bodies, A, C, impelled as above, be inverſely as their quantities of matter; theſe ſpaces are deſcribed in equal times. For if CD be not deſcribed in the same time with AB, let ſome other line as CE be the contemporary ſpace, which is greater or leſs than CD by an aſſignable magnitude EC. Then we have by the propoſition, as $AB : CE :: \frac{1}{A} : \frac{1}{C}$, and by hypotheſis, as $AB : CD :: \frac{1}{A} : \frac{1}{C}$; hence CE = CD which is abſurd; conſequently AB, CD, are deſcribed in equal times.

PROPOSITION IV.

Let PG, QI, (fig. 4. pl. A.) be two equal and ſimilar pendulums, which are ſuſpended as in prop. 1, at P, Q, the centres of gravity of the veſſels AC, *ac*; alſo let them begin to deſcend at the ſame moment from the ſituations PG, QI, in which they are equally inclined to the horizon, and parallel to each other: they will arrive at the vertical poſitions NM, *nm* in the ſame time, whatever be the weights of the two veſſels; that is, their times of vibrating through equal angles are equal.

DEMONSTRATION. Let T, t, be the centres of gravity of the ſyſtems $ACPG$, $acQI$; put the common weight of the pendulums $= b$; weight of $AC = w$; that of $ac = W$; the lines PG, $QI = l$; $NT = S$; $nt = s$: now the points T, t, will always be in the right lines MN, mn, by prop. 1; alſo $NM = PT$, and $nm = Qt$, ibid.; hence PT or $NM = \frac{bl}{b+w}$; and Qt or $nm = \frac{bl}{b+W}$ (cor. 1, prop. 1.); therefore, as $PT : Qt :: \frac{1}{b+w} : \frac{1}{b+W}$; that is, PT is to Qt inverſely as the quantities of matter $ACPG$, $acQI$, which are to be moved through the right lines TM, tm. Moreover the angles TPN, tQn, are equal by hypotheſis at the commencement of motion; therefore the motive forces acting upon T, t, in the lines NT, nt, are equal (cor. prop. 2.): hence as $\dot{S} : \dot{s} :: \frac{1}{b+w} : \frac{1}{b+W}$ (prop. 3.): but the triangles NPT, nQt, are ſimilar, being right-angled at N, n; hence alſo, as $PT : Qt :: NT : nt :: \frac{1}{b+w} : \frac{1}{b+W}$; conſequently as $PT : Qt :: S + \dot{S} : s + \dot{s}$; that is, the motive forces acting upon T, t, will continue equal between themſelves, becauſe the variable triangles, NPT, nQt, will always be ſimilar; but $NM = PT$, and $nm = Qt$; therefore as $NM : nm :: NT : nt$; hence, as $TM : tm :: \frac{1}{b+w} : \frac{1}{b+W}$. Now when PG, QI become vertical, the points T, t, coincide with M, m, after moving through the ſpaces TM, tm; theſe ſpaces are therefore deſcribed in the ſame time (cor. prop. 3.); that is, the pendulums vibrate through equal angles in equal times.

Q. E. D.

COR. If W be infinite, I the centre of gravity of QI vibrates about the fixed point Q (cor. 2. prop. 1.); conſequently the pendulum PG is ſynchronous to the pendulum QI, which being equal and ſimilar to PG has its point of ſuſpenſion fixed.

PROPOSITION V.

Let the vertical plane, CVD (fig. 6. pl. A.), repreſent a fruſtum of a ſphere, which is ſupported by a horizontal plane touching it at V; let P be the centre of the ſphere, G the centre of gravity of the fruſtum; alſo through P draw the horizontal line pR, and make GR perpendicular to RP, in which, take RK = PG: the point P will move over the ſpace PR while the point G deſcends to K; and this will happen while a ſimilar fruſtum of an equal ſphere vibrates on its ſpherical centre, through an angle equal to VPG.

Demonstration. Complete the ſpherical ſurface AVD, which will be ſupported by the reaction of the touching plane at V, exerted along VP; therefore, AVD is a veſſel without weight, upon whoſe centre of gravity P, the pendulum PG revolves; conſequently, G will deſcend from G to K, while P moves through PR; that is, while the radius PG deſcribes the angle GPV (cor. 1, prop. 1). Let now the weight of AVD become infinite, and G will deſcribe the angle GPV in the ſame time (cor. prop. 1.). *Q. E. D.*

Cor. Since the angular velocity of G about P is the ſame whether P be fixed or not: let v denote the velocity of G in the circle GL, when GR is its diſtance from PR, which velocity may be found from the fixed pendulum; then becauſe GP, GR, are perpendiculars to a tangent at G and to PR, we have as $GP : GR :: v : \frac{v \times GR}{GP}$ = contemporary velocity of P in PR.

ARTICLE III.

An ocular demonſtration of the forty-ſeventh propoſition of the firſt book of Euclid: By Mr. H. Douglas.

Let BEL be a right angled triangle, having the right angle LBE; the ſquare DELM deſcribed upon the ſide LE, is equal to the ſquare ABEF deſcribed upon the ſide BE, together with the ſquare BIQL deſcribed upon the ſide BL. Fig. 7, pl. A.

At C making BC equal to AB, draw CG parallе rallel to BE meeting FE produced in G: then BCGE is a ſquare and equal to the ſquare ABEF. Join CI, GD, and produce DE to K; draw KH parallel to LE, and MN perpendicular to CG.

In the triangles BEL, KEF; the ſide BE is equal to the ſide EF, the angle LEB to the angle KEF, and the angles at B and F are equal becauſe they are right angles, the third angles are therefore equal, and the triangles equal in all reſpects; KE, therefore, is equal to EL the ſide of the hypothenuſal ſquare, and KF to BL the ſide of the leſſer ſquare. The remaining ſpace AKEB of the ſquare AFEB is therefore equal to the ſpace LEGC; and in the triangles, AHK, CLP, the ſides AK, CL, and the angles AKH, CLP, being reſpectively equal, and the angles at A and C right angles, the triangles are equal in all reſpects; the ſpace EBHK is therefore equal to the ſpace GELP. Again, in the triangles LBE, GED, the ſide BE is equal to the ſide EG, the ſide LE to the side DE, and the angle BEL to the angle GED, the triangles are therefore identical and equal in all reſpects; the ſide GD is, therefore, equal to the ſide LB, and the angle EGD to the angle LBE, that is, the angle EGD is a right angle; therefore the points C, G, D, are in a ſtraight line.

Again, the triangle MND is equal to the triangle LBE, becauſe LE and MD are equal being ſides of the ſame ſquare, the angles at B and N are right angles, and the angle MDN equal to the angle BEL, the third angles are therefore equal and the triangles identical; MN then is equal to BL the ſide of the ſquare BIQL, and the triangle MDN equal in all reſpects to the triangle EKF.

The triangles GDE, CBI, having the ſide GE equal to the ſide CB, the ſide GD to the ſide BI (or BL), and the angles at G and B right angles, are equal in all reſpects; the angle ICB therefore is equal to the angle GED, that is, equal to the angle PLC; therefore CI is parallel to LP, and LO to CP; wherefore the figure COLP is a parallelogram, and the triangle OLC equal to the triangle CLP.

Now the triangle CBI is made up of the trapezoid IBOL and the triangle COL or the triangle CLP; the triangle GED is therefore equal to the trapezoid IBOL and the triangle CLP; or if the triangle ERS be made equal to the triangle COL (which is equal to the triangle AKH), then the trapezoid IBOL is equal to the trapezoid GRSD.

Again, the triangles PNM, IOQ, having the ſide MN equal to the ſide IQ or BL, and the angles MPN, QOI, (each equal to the oppoſite angles CPL, COL, of the parallelogram COLP) are equal, and the angles at N and Q (right angles) alſo equal, the triangles are therefore identical or equal in all reſpects.

The ſquare of the hypothenuſe then contains the trapezoid GRSD, the trapezium LEGP and the triangles RES, MND and PNM which are together equal to the ſum of the ſquares AFEB and IBLQ.

ARTICLE IV.

An inveſtigation of ſome theorems which are of uſe in obtaining the ſums of certain infinite ſeries by means of circular arcs.

By Mr. JAMES CUNLIFFE, *R. M. College.*

1. LET $\frac{1 - cx}{1 - 2cx + x^2}$ be expanded in an infinite ſeries.

Put $\frac{1 - cx}{1 - 2cx + x^2} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \&c.$

multiplying the expreſſion by $1 - 2cx + x^2$ gives

$$\begin{array}{rcccccc} 1 - cx = 1 & + Ax & + Bx^2 & + Cx^3 & + Dx^4 & + & \&c. \\ & - 2cx & - 2cAx^2 & - 2cBx^3 & - 2cCx^4 & - & \&c. \\ & & + x^2 & + Ax^3 & + Bx^4 & + & \&c. \end{array}$$

whence

$$\begin{array}{rcccccc} 0 = & Ax & + Bx^2 & + Cx^3 & + Dx^4 & + & \&c. \\ & - cx & - 2cAx^2 & - 2cBx^3 & - 2cCx^4 & - & \&c. \\ & & + x^2 & + Ax^3 & + Bx^4 & + & \&c. \end{array}$$

making the coefficients of the like powers of $x = 0$, we get $A = c$; $B = 2cA - 1$; $C = 2cB - A$; $D = 2cC - B$; &c.

Let c be the coſine of a circular arch z, radius 1; then by a known theorem $B = 2c^2 - 1 = \text{coſ. } 2z$;
$C = 2cB - c = \text{coſ. } 3z$; $D = 2cC - B = \text{coſ. } 4z$, &c.

Therefore $\frac{1 - cx}{1 - 2cx + x^2} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \&c.$

$= 1 + x \,.\, \text{coſ. } z + x^2 .\, \text{coſ. } 2z + x^3 .\, \text{coſ. } 3z + x^4 .\, \text{coſ. } 4z + \&c.$

2. Take $x = 1$, then $\frac{1 - cx}{1 - 2c + x^2} = \frac{1 - c}{2(1 - c)} =$

$$\tfrac{1}{2} = 1 + \text{coſ.}\ z + \text{coſ.}\ 2z + \text{coſ.}\ 3z + \text{coſ.}\ 4z + \&\text{c.}$$

or, $-\tfrac{1}{2} = \text{coſ.}\ z + \text{coſ.}\ 2z + \text{coſ.}\ 3z + \text{coſ.}\ 4z + \&\text{c.}$ multiplying this expreſſion by $\dot{z}$ and taking the fluents

$$-\frac{z}{2} = \text{ſin.}\ z + \tfrac{1}{2}\ \text{ſin.}\ 2z + \tfrac{1}{3}\ \text{ſin.}\ 3z + \tfrac{1}{4}\ \text{ſin.}\ 4z + \&\text{c.}$$

Put $q =$ the quadrantal arc of a circle, radius 1, then when $z = 2q$, the ſines of z, $2z$, $3z$, $4z$, &c. will be each $= 0$; therefore the correct equation of the fluents will be

$$q - \frac{z}{2} = \text{ſin.}\ z + \tfrac{1}{2}\ \text{ſin.}\ 2z + \tfrac{1}{3}\ \text{ſin.}\ 3z + \tfrac{1}{4}\ \text{ſin.}\ 4z + \&\text{c.}$$

3. Take $x = -1$, then $\frac{1 - cx}{1 - 2cx + x^2} = \frac{1 + c}{2(1 + c)} =$

$$\tfrac{1}{2} = 1 - \text{coſ.}\ z + \text{coſ.}\ 2z - \text{coſ.}\ 3z + \text{coſ.}\ 4z - \&\text{c.}$$

whence by tranſpoſition

$$\tfrac{1}{2} = \text{coſ.}\ z - \text{coſ.}\ 2z + \text{coſ.}\ 3z - \text{coſ.}\ 4z + \&\text{c.}$$

multiplying this expreſſion by $\dot{z}$ and taking the fluents

$$\frac{z}{2} = \text{ſin.}\ z - \tfrac{1}{2}\ \text{ſin.}\ 2z + \tfrac{1}{3}\ \text{ſin.}\ 3z - \tfrac{1}{4}\ \text{ſin.}\ 4z + \&\text{c.}$$

Taking half the ſum of the preceding expreſſion and that deduced at art. 2, and we ſhall have

$$\frac{q}{2} = \text{ſin.}\ z + \tfrac{1}{3}\ \text{ſin.}\ 3z + \tfrac{1}{5}\ \text{ſin.}\ 5z + \&\text{c.}$$

Alſo taking their difference

$$q - z = \text{ſin}\ 2z + \tfrac{1}{2}\ \text{ſin.}\ 4z + \tfrac{1}{3}\ \text{ſin.}\ 6z + \&\text{c.}$$

The foregoing theorems are obtained in a different manner at the beginning of Landen's 5th. Memoir.

ARTICLE V.

An inveſtigation of ſome theorems which are of uſe in obtaining the ſums of certain infinite ſeries by means of hyperbolic logarithms. By Mr. JAMES CUNLIFFE.

1. LET the expreſſion $\frac{s}{1-2cx+x^2}$ be expanded in an infinite ſeries.

Put $\frac{s}{1-2cx+x^2} = s + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + \&c.$

and multiplying both ſides of the expreſſion by $1 - 2cx + x^2$,

$$\begin{array}{rcccccc} s = s + & Ax + & Bx^2 + & Cx^3 + & Dx^4 + & Ex^5 + & \&c. \\ & - 2csx & - 2cAx^2 & - 2cBx^2 & - 2cCx^4 & - 2cDx^5 & - \&c. \\ & & + sx^2 + & Ax^3 + & Bx^4 + & Cx^5 + & \&c. \end{array}$$

making the coefficients of the homologous terms $= 0$;

$A = 2sc$; $B = 2cA - s$; $C = 2cB - A$; $D = 2Cc - B$, &c.

Now if s and c denote the ſine and coſine of a circular arch z, radius 1: then by known theorems

$A = 2cs = \text{ſin. } 2z$; $B = 2cA - s = \text{ſin. } 3z$; $C = 2cB - A = \text{ſin. } 4z$; $D = 2cC - B = \text{ſin. } 5z$, &c.

and therefore $\frac{s}{1-2cx+x} = s + Ax + Bx^2 + Cx^3 + Dx^4 + \&c.$

$= \text{ſin. } z + x . \text{ſin. } 2z + x^2 . \text{ſin. } 3z + x^3 . \text{ſin. } 4z + x^4 . \text{ſin. } 5z + \&c.$

2. Take $x = 1$, then

$$\frac{s}{1-2cx+x^2} = \frac{s}{2(1-c)} = \text{ſin.} z + \text{ſin.} 2z + \text{ſin.} 3z + \text{ſin.} 4z + \&c.$$

and it is well known that $\frac{\dot{c}}{s} = - \dot{z}$

multiplying the former expreſſion by this

$$\frac{\dot{c}}{2(1-c)} = - \dot{z} \times (\text{ſin. } z + \text{ſin. } 2z + \text{ſin. } 3z + \text{ſin. } 4z + \&c.$$

taking the fluents

$$\tfrac{1}{2} \times \text{h. l.} \left(\frac{1}{1-c}\right) = \text{coſ. } z + \tfrac{1}{2} \text{coſ. } 2z + \tfrac{1}{3} \text{coſ. } 3z + \tfrac{1}{4} \text{coſ. } 4z + \&c.$$

But when $c = 0$, then the coſines of z, $3z$, $5z$, &c. will each $= 0$; and the coſines of $2z$, $4z$, $6z$, $8z$, &c. will be -1, $+1$, -1, $+1$, &c.; therefore the correct equation

tion of the fluents will be

$$\tfrac{1}{2}\times\text{h. l.}\left(\frac{1}{1-c}\right)=\begin{cases}\text{coſ.}\,z+\tfrac{1}{2}\text{coſ.}\,2z+\tfrac{1}{3}\text{coſ.}\,3z+\tfrac{1}{4}\text{coſ.}\,4z+\&\text{c.}\\ +\tfrac{1}{2}-\tfrac{1}{4}+\tfrac{1}{6}-\tfrac{1}{8}+\&\text{c.}\end{cases}$$

$$=\begin{cases}\text{coſ.}\,z+\tfrac{1}{2}\text{coſ.}\,2z+\tfrac{1}{3}\text{coſ.}\,3z+\tfrac{1}{4}\text{coſ.}\,4z+\&\text{c.}\\ +\tfrac{1}{2}\times\text{h. l. }2;\end{cases}$$

whence, by transpoſition,

$$\tfrac{1}{2}\times\text{h. l.}\left(\frac{1}{1-c}\right)-\tfrac{1}{2}\times\text{h. l. }2=\tfrac{1}{2}\times\text{h. l. }\frac{1}{2\,(1-c)}=$$
$$\text{coſ.}\,z+\tfrac{1}{2}\,\text{coſ.}\,2z+\tfrac{1}{3}\,\text{coſ.}\,3z+\tfrac{1}{4}\,\text{coſ.}\,4z+\&\text{c.}$$

3. Again, take $x=-1$, then,

$$\frac{s}{1-2cx+x^2}=\frac{s}{2(1+c}=\text{ſin.}\,z-\text{ſin.}\,2z+\text{ſin.}\,3z-\text{ſin.}\,4z+\&\text{c.}$$

multiplying this expreſſion by $\frac{\dot{c}}{s}=-\dot{z}$, gives

$$\frac{\dot{c}}{2(1+c)}=-\dot{z}\times(\text{ſin.}\,z-\text{ſin.}\,2z+\text{ſin.}\,3z-\text{ſin.}\,4z+\&\text{c.})$$

and taking the fluents

$$\tfrac{1}{2}\times\text{h. l. }(1+c)=\text{coſ.}\,z-\tfrac{1}{2}\,\text{coſ.}\,2z+\tfrac{1}{3}\,\text{coſ.}\,3z-\tfrac{1}{4}\,\text{coſ.}\,4z+\&\text{c.}$$

But when $c=0$, the coſines of z, $3z$, $5z$, &c. will be each $=0$; and the coſines of $2z$, $4z$, $6z$, $8z$, &c. will be -1, $+1$, -1, $+1$, &c.; wherefore the correct equation of the fluents will be

$$\tfrac{1}{2}\times\text{h. l. }(1+c)=\begin{cases}\text{coſ.}\,z-\tfrac{1}{2}\,\text{coſ.}\,2z+\tfrac{1}{3}\,\text{coſ.}\,3z-\tfrac{1}{4}\,\text{coſ.}\,4z+\&\text{c.}\\ -\tfrac{1}{2}+\tfrac{1}{4}-\tfrac{1}{6}+\tfrac{1}{8}-\&\text{c.}\end{cases}$$

$$=\begin{cases}\text{coſ.}\,z-\tfrac{1}{2}\text{coſ.}\,2z+\tfrac{1}{3}\text{coſ.}\,3z-\tfrac{1}{4}\,\text{coſ.}\,4z+\&\text{c.}\\ -\tfrac{1}{2}\times\text{h. l. }2;\end{cases}$$

therefore, by tranſpoſition,

$$\tfrac{1}{2}\times\text{h. l. }(1+c)+\tfrac{1}{2}\times\text{h. l. }2=\tfrac{1}{2}\times\text{h. l. }\left\{2\,(1+c)\right\}=$$
$$\text{coſ.}\,z-\tfrac{1}{2}\,\text{coſ.}\,2z+\tfrac{1}{3}\,\text{coſ.}\,3z-\tfrac{1}{4}\,\text{coſ.}\,4z+\&\text{c.}$$

And from what has been deduced before

$$\tfrac{1}{2}\times\text{h. l. }\frac{1}{2\times(1-c)}=\text{coſ.}\,z+\tfrac{1}{2}\text{coſ.}\,2z+\tfrac{1}{3}\text{coſ.}\,3z+\tfrac{1}{4}\text{coſ.}\,4z+\&\text{c.}$$

half the ſum of theſe two expreſſions, by the nature of logarithms, is

$$\tfrac{1}{4}\times\text{h. l.}\left(\frac{1+c}{1+c}\right)=\text{coſ.}\,z+\tfrac{1}{3}\,\text{coſ.}\,3z+\tfrac{1}{5}\,\text{coſ.}\,5z+\&\text{c.}:$$

alſo, taking their difference,

$$\tfrac{1}{2}\times\text{h. l. }\frac{1}{4(1-c^2)}=\text{h. l. }\frac{1}{2\sqrt{(1-c^2)}}=\text{coſ.}\,2z+\tfrac{1}{2}\,\text{coſ.}\,4z+\tfrac{1}{3}\,\text{coſ.}\,6z+\&\text{c.}$$

The above theorems are obtained by a method materially different in Mr. Landen's 5th. Memoir, Art. 16.

ARTICLE VI.

Solution of a Dynamical Queſtion. By Mr. JOHN BARRY.

To the Editor of the Mathematical Repoſitory.

SIR,

In Art. 6, part 2, of the firſt vol. of the Repoſitory, there is an ingenious ſolution by Mr. Ivory of a problem in dynamics, firſt propoſed by Mr. Simpſon in his Miſcellaneous Tracts, and afterwards introduced by Mr. Atwood in his Treatiſe on Rectilineal Motion, pa. 289, as a particular caſe of a general propoſition. This gentleman's ſolution differing ſo widely from Mr. Simpſon's, induced me to conſider his inveſtigation; and I find that his error proceeds from a wrong application of his own general theorem in aſcertaining the ratio of the ſecond fluxions of the quantities x and y.

Should you think the following ſolution of Mr. Atwood's general propoſition (pa. 285, of his ſaid treatiſe on rectilineal motion), from which I have eaſily deduced a ſolution to Mr. Simpſon's problem, deſerving of a place in your valuable work, I requeſt you will do me the favour to publiſh it.

I am, Sir, your's, &c.

Limerick, JOHN BARRY.
July 29, 1806.

PROPOSITION, *by Mr.* ATWOOD.

Let AFC, BGC (fig. 8, pl. A.), be two curves, the planes of which are vertical; and let a line, ACB, be ſtretched over a fixed pully, C, by two given weights A and B, of which B preponderates againſt A, and deſcends along the curve CGB: it is required to aſſign the velocity of B, when it has deſcended through a given perpendicular altitude, and A has aſcended through a given altitude in the ſame time?

SOLUTION, *by Mr.* BARRY.

According to Mr. Atwood, let BL be the perpendicular altitude, through which B has deſcended, and let AH be the perpendicular altitude through which A has aſcended in the ſame time; and let BL $= y$, AH $= q$. Suppoſe the weight B to deſcribe in its deſcent the evaneſcent arc bB, and during the ſame time let the other weight A riſe through the arc aA. Through b and A draw bo and Al parallel to the horizon; alſo through b draw bn perpendicular to CB, and Am perpendicular to Ca, and let CB $= x$, Bn $= \dot{x} =$ Am, Bo $= \dot{y}$, Bb $= \dot{s}$, Aa $= \dot{r}$, and al $= \dot{q}$.

Now Mr. Atwood has truly found in his ſaid treatiſe, pa. 286, that the whole force by which B is urged in the direction bB, is $= \frac{B\dot{y} - A\dot{q}}{\dot{s}}$, and he has alſo ſhewn in page 287 that the equivalent maſs, which being accumulated in B, when A is removed, will reſiſt the communication of motion in the ſame manner as the maſs A aſcending by the force of B is $= \frac{A\dot{r}\ddot{r}}{\dot{s}\ddot{s}}$.

Let $v =$ the velocity of the deſcending weight B, $w =$ the velocity of the aſcending weight A, $t =$ the correſponding time, and $b =$ 193 inches.

Then, by the laws of motion, $\dot{s} = v \times \dot{t}$ and $\dot{r} = w \times \dot{t}$; and ſince the evaneſcent arcs aA and bB are deſcribed in the ſame time, we have $\ddot{s} = \dot{v} \times \dot{t}$ and $\ddot{r} = \dot{w} \times \dot{t}$; therefore, by ſubſtitution, $\frac{A\dot{r}\ddot{r}}{\dot{s}\ddot{s}} = \frac{Aw\dot{w}}{v\dot{v}}$. Conſequently, the force which accelerates B is

$$= \frac{B\dot{y} - A\dot{q}}{\dot{s}} \div \left(B + \frac{Aw\dot{w}}{v\dot{v}}\right) = \frac{(B\dot{y} - A\dot{q})\, v\dot{v}}{\dot{s}\,(Bv\dot{v} + Aw\dot{w})}.$$

Hence, by the laws of motion,

$$v\dot{v} = b \times \dot{s} \times \frac{(B\dot{y} - A\dot{q})\, v\dot{v}}{\dot{s}\,(Bv\dot{v} + Aw\dot{w})} = b \cdot \times \frac{(B\dot{y} - A\dot{q})\, v\dot{v}}{Bv\dot{v} + Aw\dot{w}}.$$

Therefore $b\,(B\dot{y} - A\dot{q}) = Bv\dot{v} + Aw\dot{w}$, and taking the fluents

$b\,(By - Aq) = \frac{Bv^2}{2} + \frac{Aw^2}{2} - d$, d being the neceſſary correction.

Hence $v^2 = 2by - \frac{2bAq}{B} - \frac{Aw^2}{B} + 2d$.

It is obvious that this equation is general, whether the bodies move in right or curved lines.

Cor. It is evident that $v : w :: Bb : Aa :: \dot{s} : \dot{r}$; hence $w^2 = v^2\dot{r}^2 \div \dot{s}^2$. Therefore, by ſubſtitution, we get

$$v^2 = 2by - \frac{2bAq}{B} - \frac{Av^2\dot{r}^2}{B\dot{s}^2} + 2d.$$ And therefore

$$v = \dot{s}\sqrt{\frac{2b\,(By - Aq) + 2dB}{B\dot{s}^2 + A\dot{r}^2}}.$$

In order to apply this general expreſſion for the velocity of B to the ſolution of Mr. Simpſon's problem, which Mr. Atwood expreſſes in the following manner, ſee pa. 289 of the treatiſe on Rectilineal Motion.

Let two equal weights, A, A, (fig. 9, pl. A.) be faſtened to the extremities of a line which goes over the fixed points E and C, which are horizontal: when the line AECA is ſtretched by the equal weights A, A, let a body D be fixed to the middle point D, it will deſcend from reſt at D in the direction DB, perpendicular to the horizon, at the ſame time elevating the weights A, A. Suppoſe it were required to aſſign the velocity of the deſcending weight B, when it has deſcribed any ſpace DB.

Let ED $= a$, EB $= y$. and DB $= x$. Then by ſubſtituting in the expreſſion for the velocity in the corollary, x for y, y for q, $\dot{x}$ for $\dot{s}$, and $\dot{y}$ for $\dot{r}$, we have v, or the velocity of B $= \dot{x}\sqrt{\dfrac{2b\,(Bx - 2Ay) + 2dB}{B\dot{x}^2 + 2A\dot{y}^2}}$, where 2A is written for A, becauſe each of the weights A, A, acts in a ſimilar manner on B. But in the preſent caſe $\dot{y} = \dfrac{x\dot{x}}{y}$. Hence, by ſubſtitution, $v = y\sqrt{\dfrac{2b\,(Bx - 2Ay) + 2dB}{By^2 + 2Ax^2}}$. Let $B \div 2A = m$, and put $y^2 - a^2$ for its equal x^2, then we have

$$v = y\sqrt{\frac{2bmx - 2by + 2dm}{(m+1)\,y^2 - a^2}},$$ which correſponds exactly with Simpſon's expreſſion for the velocity.

Since the motion commences from the point D, then v being $= 0$, when $x = 0$ and $y = a$, we find $d = \dfrac{ba}{m} - \dfrac{2baA}{B}$, in which caſe,

$$v^2 = 2b\,(a^2 + x^2) \times \frac{Bx - 2A\sqrt{(a^2 + x^2)} + 2aA}{B\,(a^2 + x^2) + 2Ax^2}.$$

If this value of v^2 be put $= 0$, we get $x = \frac{4aAB}{4A^2-B^2} =$ the greateſt diſtance through which the weight B can deſcend before its whole motion is deſtroyed by the weights A, A.

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ARTICLE VII.

Some properties of parallelograms, with the application of them to the moments of forces. By Mr. John Gough.

To the Editor of the Mathematical Repoſitory.

Sir,

Mr. Gregory has given a beautiful theorem reſpecting the moments of forces, in his Mechanics; but the analytical plan of his work having induced him to derive his demonſtration from the arithmetic of lines, I flatter myſelf the Engliſh reader will not be diſpleaſed to ſee this intereſting propoſition in the geometrical dreſs, which I have endeavoured to give it, in the following eſſay.

I am, &c.

Middleſhaw,
August 22*d*, 1806.

John Gough.

PROPOSITION I.

Let ABCD (fig. 10, pl. A.) be a parallelogram, and BD a diameter of it; take any point T in BD, or in BD produced, and join AT, CT. The triangle ATB ſhall be equal to the triangle CTB, and the triangle ATD to the triangle CTD.

Draw the diameter AC, which will be biſected by the diameter BD in the point M. Therefore, the triangle ATM is equal to the triangle CTM, the triangle ABM to CBM, and the triangle ADM to CDM; for they have equal baſes, two and two, and the ſame altitudes (Eu. i. vi.). Therefore, by adding equals to equals, the triangle ATB is equal to the triangle CTB; and, by taking equals from equals, the triangle ATD is equal to the triangle CTD.

PROPOSITION

Sc, Sd, Se, Sf, Sr, &c. perpendicular to theſe lines reſpectively. Then the moment $Sr \times AR$ will be equal to the exceſs of the ſum of the moments $Sb \times AB$, $Sc \times AC$, $Sd \times AD$, lying on the ſide of AR contrary to S, above the ſum of the moments $Se \times AE$, $Sf \times AF$, lying on the ſame ſide of AR with S.

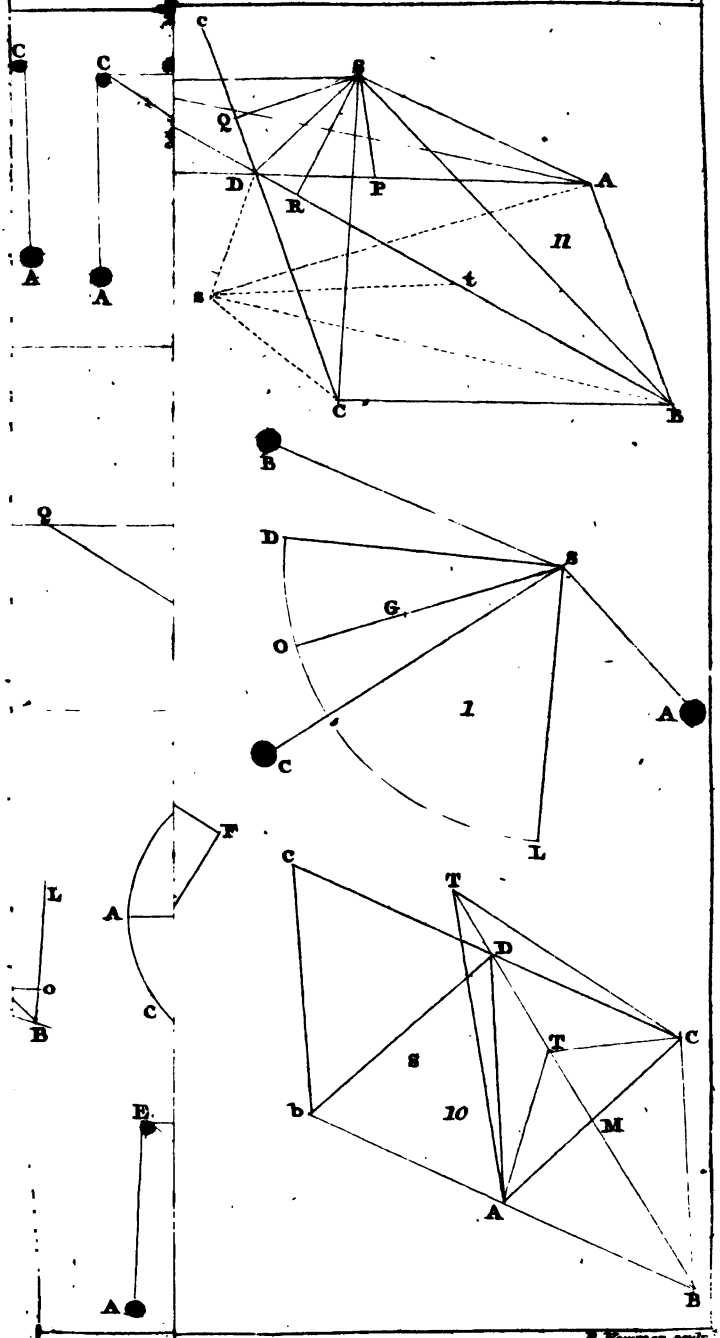
Let AG be the equivalent of AB, AC; AK the equivalent of AG, AD, that is of AB, AC, AD; and AH the equivalent of AE, AF: then ſince AR is the diameter of a parallelogram under AK, AH, and S is in the angle RAH, $Sr \times AR = Sk \times AK - Sh \times AH$ (prop. ii. and cor.): But $Sk \times AK = Sg \times AG + Sd \times AD$ (prop. ii.); for the ſame reaſon $Sg \times AG = Sb \times AB + Sc \times AC$, and $Sh \times AH = Se \times AE + Sf \times AF$; therefore $Sr \times AR = Sk \times AK - Sh \times AH = Sb \times AB + Sc \times AC + Sd \times AD - (Se \times AE + Sf \times AF)$, that is, the moment $Sr \times AR$ is equal to the exceſs of the ſum of the moments $Sb \times AB$, $Sc \times AC$, $Sd \times AD$, lying on the ſide of AR contrary to S, above the ſum of the moments $Se \times AE$, $Sf \times AF$, lying on the ſame ſide of R with S. And what has been proved of five forces may be proved in like manner of any other number.

Cor. 1. If all the component forces BA, CB, DA, and their equivalent KA, be ſo diſpoſed that the point S ſhall not be ſituated in the angle KAD or that vertically oppoſite to it, the moment of the equivalent KA is equal to the ſum of the moments of the components. This is proved in the demonſtration.

Cor. 2. If ſome of the forces in a propoſed ſcheme be negative, they are to be converted into affirmative ones by the preceding ſcholium.

ARTICLE

Vol. II. Plate A. Fig. 1 to 14.



R. Newman sculp.

ARTICLE VIII.

To the Editor of the Mathematical Repoſitory.

SIR,

The ſolutions of the following problems being conſiderably facilitated by making uſe of the mechanical propoſition demonſtrated in No. vi. (art. i. part ii. of the preſent volume) of your Repoſitory, by inſerting them, when it is convenient, you will oblige, Sir, Your's, &c.

A. B.

PROBLEM I.

If a body be connected with a horizontal plane by any point in it, and in ſuch a manner, that while it deſcends by the force of gravity, the connected point may ſlide freely along the plane without friction or any obſtruction whatever: it is required, for any poſition of the body, to inveſtigate general expreſſions for the velocity of the connected point, and for the velocities of the centres of gravity and oſcillation, as likewiſe for its preſſure upon the plane.

Fig. 15, pl. B. Let QBD be the horizontal plane, Q, the point of contact at the commencement of the motion of the body, and G and P the centres of gravity and oſcillation, q, g, p their poſition at any other time. Then (by prop. 1ſt. art. 2d. part 2d. of the preſent vol.) the point G will deſcribe the ſtraight line MGB, and the point P, the arc of an ellipſe. Suppoſe the whole body, W, to be divided into two parts P and Q, and in ſuch a manner, that P : Q :: QG : GP, then if the part Q be placed upon the plane at Q and connected with P by an inflexible line without weight, it is evident, that theſe two bodies, thus circumſtanced, will move in the ſame manner, and with the ſame velocities as the correſponding points of the given body. For the diſtances of the centres of gravity and oſcillation from Q, as well as the inclination of the line paſſing through them, to the horizontal plane, are the ſame in both caſes; wherefore the above mentioned velocities muſt be equal.

Let $QP = L$, $QG = a$, $BG = d$, $Gg = x$, $qB = y$; v, v' and V the velocities, reſpectively, of Q, G and P, when they arrive at q, g, and p, and $l = 16\frac{1}{12}$ feet, It is well known that the velocities of q and g, are to each other as $Bg : qB$, hence

$v : v' :: d - x : y$, therefore $v = v' \times \frac{d-x}{y}$. Let fall the perpendiculars PC, pc, then by the propoſition $4l \times (CP - cp) \times P = Qv^2 + PV^2$. Let pn be a ſmall particle of the curve deſcribed by p, in a given moment of time; reſolve it into pa and na, perpendicular and parallel to QC, then theſe lines will be as the velocities of p, in their reſpective directions. By the property of the centre of gravity $\frac{P+Q}{P} . v'$ = velocity of P in the the direction pa, and $\frac{Q}{P} . v = \frac{Q}{P} . v' \times \frac{d-x}{y}$ = its velocity in the direction na; hence

$$\frac{(P+Q)^2}{P^2} . v'^2 + \frac{Q^2}{P^2} . v'^2 . \frac{(d-x)^2}{y^2} = \frac{v'^2}{P^2} \times \left\{ (P+Q)^2 + Q^2 . \frac{(d-x)^2}{y^2} \right\} = V^2,$$

conſequently $4l\,(CP - cp) = 4l . \frac{Lx}{a} . P = Q . v'^2 \times \frac{(d-x)^2}{y^2} + \frac{v'^2}{P} \times \left\{ (P+Q)^2 + Q^2 \times \frac{(d-x)^2}{y^2} \right\}$. But $P : Q :: a : L - a$, therefore $Q = \frac{L-a}{a} . P$, and $P + Q = \frac{LP}{a}$, whence, by ſubſtitution and reduction, it will eaſily appear that $v'^2 = 4lx \times \frac{ay^2}{(L-a)(d-x)^2 + Ly^2}$. For y^2, in the denominator, ſubſtitute its value $a^2 - (d-x)^2$, and we have $v'^2 = 4lx . \frac{y^2}{La - (d-x)^2}$, and therefore $v' = 2\sqrt{lx} . \frac{y}{\sqrt{La - (d-x)^2}}$, hence $v = \frac{d-x}{y} . v' = 2\sqrt{lx} . \frac{d-x}{\sqrt{La - (d-x)^2}}$, and $V = 2\sqrt{lx} . \sqrt{\frac{L^2 a - (2L - a) - (d-x)^2}{a\,(La - (d-x)^2)}}$.

Put $PC = h$, then $L : a :: h : d = \frac{ah}{L}$, if therefore $x = d$, or the body has arrived at the ſituation q'BD, then $v = 2\sqrt{l \times \frac{ah}{L}} \times \sqrt{\frac{L^2 a}{a^2 L}} = 2\sqrt{lh}$, the ſame velocity which a body would acquire in deſcending freely through the ſpace PC. But when $x = d$, $y = a$, therefore, when the body is in the above mentioned ſituation, $v' = 2\sqrt{l \times \frac{ah}{L} \times \frac{a}{\sqrt{la}}} = 2\sqrt{lh} \times \frac{a}{L}$, and it is evident then that $v = 0$.

When x is very ſmall or evaneſcent, let $y = e$, then we have $v'^2 = 4lx \cdot \frac{e^2}{La - d^2}$; but when the body and ſpace deſcended are given, the accelerating force varies as the ſquare of the velocity generated, wherefore $4lx : 4lx \cdot \frac{e^2}{La - d^2} :: W\,(P + Q) : W \times \frac{e^2}{La - d^2} =$ that part of the weight of W which is not ſupported; hence, $W \times \left(s - \frac{e^2}{La - d^2}\right) =$ preſſure upon the plane.

Cor. Let s and c = ſine and coſine of the angle QGB for any poſition of the body whatever, then $y = sa$, and $d - x = ca$, rad. $= 1$, wherefore, by ſubſtitution, $v' = 2\sqrt{(lx)} \times \frac{sa}{\sqrt{La - c^2a^2}} = 2\sqrt{(lx)} \times s\sqrt{\frac{a}{L - c^2a}}$, $v = 2\sqrt{(lx)} \times \frac{ca}{\sqrt{(La - c^2a^2}} = 2\sqrt{(lx)} \times c\sqrt{\frac{c}{L - c^2a}}$ and $V = 2\sqrt{(lx)} \times \sqrt{\frac{L^2 - (2L - a) \cdot c^2a}{La - c^2a}}$. In the very ſame manner it will appear, that the preſſure upon the plane $= W \times \left(1 - \frac{s^2a^2}{La - c^2a^2}\right) = W \times \left(1 - \frac{s^2a}{L - c^2a}\right) = W \times \left(1 - \frac{s^2a}{(L - a) - s^2a}\right)$. It is evident that when the body deſcends below the plane, c, will be negative, or the point Q will move in a contrary direction to what it did before.

Ex. 1. Suppoſe the body to be a ſlender cylinder or rod; then if R = its length, it is well known that $a = \frac{1}{2}R$ and $L = \frac{2}{3}R$, therefore $a = \frac{3}{4}L$; hence, by ſubſtituting for a, its value, in the cor. it will eaſily appear that $v' = 2\sqrt{(lx)} \cdot s\sqrt{\frac{3}{4 - 3c^2}}$, $v = 2\sqrt{(lx)} \cdot c\sqrt{\frac{3}{4 - 3c^2}}$ and $V = 2\sqrt{(lx)} \times \sqrt{\frac{16 - 15c^2}{12 - 9c^2}}$. Hence when the rod arrives at the plane, and conſequently,

$s = 1$,

$s = 1$; $c = 0$; and $d = x$, we have $v' = 2\sqrt{(ld)} \times \sqrt{\tfrac{3}{4}} = \sqrt{(3ld)}$, $v = 0$, and $V = 2\sqrt{(ld)} \times \sqrt{\tfrac{4}{3}} = 4\sqrt{\frac{ld}{3}}$. Likewiſe the preſſure upon the plane $= W \times \left(1 - \frac{s^2a}{L - c^2}\right) = W \times \left(1 - \frac{\frac{3}{4}s^2L}{L - \frac{3}{4}c^2L}\right) = W \times \left(1 - \frac{3s^2}{4 - 3c^2}\right) = W \times \left(1 - \frac{3s^2}{1 + 3s^2}\right) = W \times \frac{1}{4}$ when $s = 1$ and $c = 0$.

Ex. 2. Prize Queſtion, Gentleman's Diary, 1791.

If a given ſegment of a globe greater than an hemiſphere touching an horizontal plane perfectly poliſhed, with its ſurface at the edge of its ſection or baſe, be thus put in motion by the force of uniform gravity; how far will it move along the plane, what is its velocity, and with what weight does it preſs againſt the plane at every point during the time of one titubation?

Let SQPg (fig. 16, pl. B.) repreſent the poſition of the ſegment when its motion commences, SP, QT, two diameters of the globe perpendicular to and parallel to the horizon; draw RC perpendicular to Sg and produce it to O, the centre of oſcillation of the ſegment, when C, the centre of the globe, is the point of ſuſpenſion. Let G be the centre of gravity of the ſame ſegment, and through G, draw Ga perpendicular to the horizon, it is evident that the point G will deſcend in this line, and that the point C will move in QT.

Put $SC = r$, $CR = b$, $CG = a$, $Ga = d$ and $CO = L$. By ſimilar triangles we have, $r : b :: a : d = \frac{ab}{r}$; hence, by cor. $v' = 2\sqrt{(lx)} \times s\sqrt{\frac{a}{L - c^2a}}$, $v = 2\sqrt{(lx)} \times c\sqrt{\frac{a}{L - c^2a}}$ and $V = 2\sqrt{(lx)} \times \sqrt{\frac{L^2 - (2L - a) . c^2a}{a . (L - c^2a)}}$; likewiſe the preſſure upon the plane $= W \times \left(1 - \frac{s^2a}{L - c^2a}\right)$.

When G comes to a, then $s = 1$, $c = 0$, and $x = d = \frac{ab}{r}$, therefore in that poſition, $v' = 2\sqrt{\left(l \times \frac{ab}{r}\right)} \times \sqrt{\frac{a}{L}} = 2a\sqrt{\frac{lb}{rL}}$, $v = 0$, $V = 2\sqrt{\left(l \times \frac{ab}{r}\right)} \times \sqrt{\frac{L}{a}} = 2\sqrt{\frac{lLb}{r}}$, and the preſſure $= W \times \left(1 - \frac{a}{L}\right)$.

When C arrives at a, in that poſition we ſhall have $s = 0$, $c = 1$, $x = a + \frac{ab}{r} = ar(r + b)$, and then it will appear that $v' = 0$, $v = 2\sqrt{\left\{l \cdot \frac{a}{r} \cdot (r + b)\right\}} \times \sqrt{\frac{a}{L - a}} = 2a\sqrt{\frac{l(r + b)}{r(L - a)}}$, $V = 2\sqrt{\left\{l \cdot \frac{a}{r} \cdot (r + b)\right\}} \times \sqrt{\frac{L^2 - (2L - a) \cdot a}{(L - a) \cdot a}}$, and the preſſure upon the plane $= W \times 1$. The values of a and L are eaſily found by the uſual methods of calculation.

It may be obſerved, that when N comes to P, or its greateſt altitude; the rotatory motion of the ſegment will be the greateſt poſſible, but it will continue to move the ſame way till it arrives at N′, being at the ſame diſtance from the horizontal plane as when it begun, when it will entirely ceaſe to move; it will then begin to move the contrary way, and ſo continue to vibrate for ever. The ſpace paſſed over by the centre of the globe, forwards and backwards, in one titubation is evidently $= 2CG$, the greateſt diſtance on each ſide of a, being $= CG$.

In the very ſame manner, by a proper ſubſtitution for L and a, the values of v', v and V may be found for any body whatever; as likewiſe its preſſure upon the plane, its poſition, or the values of s and c, being given.

PROBLEM II.

Prop. xxiv. ſect. vi. Atwood on Rectilinear and Rotatory Motion.

Let A, B (fig. 17, pl. B.), be a ſingle moveable and a fixed pully, by means of which the power P elevates the weight W: having given P and W, together with the weight of the cylindrical pullies, A and B, it is required to aſſign the ſpace which the deſcending weight P deſcribes in a given time, the weight of the moveable pully being included in the weight W.

Let $v =$ velocity of P, then $\frac{v}{2} =$ velocity of W, $\frac{v}{\sqrt{2}} =$ velocity of the centre of gyration of the pully A, and $\frac{v}{2\sqrt{2}} =$ velocity of the ſame centre in the pully B. Let $s =$ ſpace deſcended by P in the time t; then we have the following equation, *viz.*

$$4l\left(Ps - W \cdot \frac{s}{2}\right) = Pv^2 + W \cdot \frac{v^2}{4} + Q \cdot \left(\frac{v^2}{2} + \frac{v^2}{8}\right) = Pv^2 + W \cdot \frac{v^2}{4} + Q \cdot \frac{5v^2}{8},$$

but $v = \frac{2s}{t}$, therefore $4l\left(Ps - W.\frac{s}{2}\right) = \frac{4P^2s^2}{t^2} + W.\frac{s^2}{t^2} + Q.\frac{5s^2}{2t^2}$, therefore $s = \frac{4lt^2 \times (2P - W)}{8P + 2W + 5Q}$.

N.B. Q equal the weight of each pully.

PROBLEM III.

Prop. xxv. ſect. vi. Atwood's Treatiſe on Motion.

In a ſyſtem of pullies in which the ſame ſtring goes round all the pullies contained in two blocks; having given the power P and the weight W raiſed by it, together with the number of the pullies and the weight and figure of each, it is required to aſſign the force which accelerates the deſcent of P, the weight of the lower block being included in the weight W.

Let t = any time, v = velocity generated in P in that time, and s = ſpace deſcended; then by mechanics $\frac{v}{n}$ = velocity of W. The velocity of the circumference of the loweſt pully = that of W = $\frac{v}{n}$, the velocity of the circumference of the next loweſt = $\frac{2v}{n}$, of the next, = $\frac{3v}{n}$, &c. to $\frac{nv}{n}$; and the velocities of their reſpective centres of gyration = $\frac{v}{n\sqrt{2}}, \frac{2v}{n\sqrt{2}}, \frac{3v}{n\sqrt{2}} \ldots \frac{nv}{n\sqrt{2}}$. Hence this equation $4l\left(Ps - W.\frac{s}{n}\right) = Pv^2 + W.\frac{v^2}{n^2} + Q.\left\{\frac{v^2}{2n^2} \times (1 + 4 + 9 \ldots\ldots n^2)\right\}$. For s, ſubſtitute $\frac{vt}{2}$, then $2lt \times (12Pn^2 - 12Wn) = \left\{12Pn^2 + 12W + Q \times (2n^3 + 3n^2 + n)\right\} \times v$; therefore $v = 2lt \times \frac{12Pn^2 - 12Wn}{12Pn^2 + 12W + Q.(2n^3 + 3n^2 + n)}$,

Make $t = 1$, then this expreſſion will repreſent the force which accelerates P, when the force of gravity = $2l$.

N.B. The ſum of $1 + 4 + 9 \ldots n^2 = \frac{1}{6}(2n^3 + 3n^2 + n)$.
Q = the weight of each pully, which are ſuppoſed cylindrical.

ARTICLE

ERRATA.

In page 4, vol. 2, part 2, line 11, from the bottom,
for $A \times \frac{x^2u^2}{y^2} + D \times \frac{x^2u^2}{z^2}$, *read* $A \times \frac{x^2u^2}{y^2} + Bu^2 + D \times \frac{x^2u^2}{z^2}$

ARTICLE IX.

A DIOPHANTINE PROBLEM. *By Mr.* CUNLIFFE,

To find values for the ſides of a triangle in rational numbers ſuch, that the lengths of three right lines from the angles to the middle of the oppoſite ſides may be expreſſed by rational numbers.

SOLUTION.

Let ACB (fig. 18, pl. B.) repreſent the triangle, CD, AE and BF right lines from the angles to the middle of the oppoſite ſides: put AD = DB = x, AF = FC = y, and BE = EC = z. Then by a known property of triangles $AC^2 + BC^2 - 2AD^2 = 2CD^2$, or $\frac{1}{2}(AC^2 + BC^2) - AD^2 = 2y^2 + 2z^2 - x^2 = CD^2 =$ a ſquare. In like manner $2x^2 + 2z^2 - y^2 = BF^2 =$ a ſquare; and $2x^2 + 2y^2 - z^2 = AE^2 =$ a ſquare.

Put $x + y - n = z$, then $2x^2 + 2y^2 - z^2 = 2x^2 + 2y^2 - (x + y - n)^2 = x^2 - 2xy + y^2 + 2n(x + y) - n^2 =$ a ſquare; and $2x^2 + 2z^2 - y^2 = 2x^2 + 2(x + y - n)^2 - y^2 = 4x^2 + 4xy + y^2 - 4n(x + y) + 2n^2 =$ a ſquare.

Aſſume $x - y + s$ for the root of the firſt, and $2x + y - r$ for the root of the ſecond of the preceding expreſſions: then, $x^2 - 2xy + y^2 + 2n(x + y) - n^2 = (x - y + s)^2 = x^2 - 2xy + y^2 + 2s(x - y) + s^2$; and $4x^2 + 4xy + y^2 - 4n(x + y) + 2n^2 = (2x + y - r)^2 = 4x^2 + 4xy + y^2 - 2r(2x + y) + r^2$; from the former of theſe

$$x = \frac{n^2 + s^2 - 2y(n + s)}{2(n - s)} = \frac{2n^2 + 2s^2 - 4y(n + s)}{4(n - s)},$$

and from the latter $x = \dfrac{r^2 - 2n^2 + 2y(2n - r)}{4(r - n)}$;

therefore $\dfrac{2n^2 + 2s^2 - 4y(n + s)}{4(n - s)} = \dfrac{r^2 - 2n^2 + 2y(2n - r)}{4(r - n)}$, making the numerators and denominators equal to each other, $2n^2 + 2s^2 - 4y(n + s) = r^2 - 2n^2 + 2y(2n - r)$, from whence $y = \dfrac{4n^2 + 2s^2 - r^2}{8n + 4s - 2r}$. And $4(n - s) =$

4$(r - n)$,

$4(r - n)$; whence $n = \frac{1}{2}(r + s)$, by means of which $y = \frac{6s^2 + 4rs}{4(r + 4s)}$; $x = \frac{n^2 + s^2 - 2y(n + s)}{2(n - s)} = \frac{r^2 + 3rs - 2s^2}{4(r + 4s)}$; and $z = x + y - n = \frac{-(4s^2 + 3rs + r^2}{4(r + 4s)}$.

Now as only the ſquares of x, y, and z are concerned in the problem, it is manifeſt that either poſitive or negative values of theſe quantities may be taken; wherefore, rejecting the common denominator $4(r + 4s)$, we may put $x = r^2 + 3rs - 2s^2$, $y = 6s^2 + 4rs$, and $z = r^2 + 3rs + 4s^2$, and theſe being written in the remaining expreſſion to be made a ſquare, *viz.* $2y^2 + 2z^2 - x^2$ it becomes $r^4 + 6r^3s + 61r^2s^2 + 156rs^3 + 100s^4$.

Aſſume $r^2 - \frac{39rs}{5} - 10s^2$ for its root, that is, put

$$r^4 + 6r^3s + 61r^2s^2 + 156rs^3 + 100s^4 = \left(r^2 - \frac{39rs}{5} - 10s^2\right)^2$$
$$= r^4 - \frac{78r^3s}{5} + \left(\tfrac{39}{5}\right)^2 \times r^2s^2 - 20r^2s^2 + 156rs^3 + 100s^4;$$

which, after proper reduction gives, $r = -\frac{14s}{15}$.

Take $s = 15$, then $r = -14$, $x = -884$, $y = 510$ and $z = 466$: and as theſe values are all even numbers, they will expreſs the ſides of a plane triangle that will anſwer.

Let us now examine the reſult of a general method of proceeding. In the preceding part of the ſolution we obtained the equation

$$\frac{2n^2 + 2s^2 - 4y(n + s)}{4(n - s)} = \frac{r^2 - 2n^2 + 2y(2n - r)}{4(r - n)};$$

from whence we have $y = \frac{-2n^2(r - s) + n(r^2 + 2s^2) - rs(r + 2s)}{2\left\{n(4s - r) - 3rs\right\}}$,

and by means of this

$$x = \frac{r^2 - 2n^2 + 2y(2n - r)}{4(r - n)} = \frac{n^2(r + 2s) - n(r^2 + 2s^2) - rs(r - s)}{2\left\{n(4s - r) - 3rs\right\}};$$

and $z = x + y - n = \frac{-n^2(4s - r) + 6rsn - rs(2r + s)}{2\left\{n(4s - r) - 3rs\right\}}$.

And as only the ſquares of x, y, and z are concerned in the problem, it is evident that either poſitive or negative values of theſe quantities may be taken; therefore rejecting the common denominator $2\left\{n(4s - r) - 3rs\right\}$ we may take

$$x = n^2(r + 2s) - n(r^2 + 2s^2) - rs(r - s),$$
$$y = 2n^2(r - s) - n(r^2 + 2s^2) + rs(r + 2s),$$
$$z = n^2(4s - r) - 6rsn + rs(2r + s);$$

and theſe values being written in the remaining expreſſion to be made a ſquare, *viz.* $2y^2 + 2z^2 - x^2$, it becomes $9n^4(2s-r)^2 + 2n^3(12s^3 - 54s^2r + 18sr^2 - 3r^3) + n^2(r^4 - 34r^3s + 150r^2s^2 + 68rs^3 + 4s^4) + 2n(3r^4s - 21r^3s^2 - 6r^2s^3 + 6rs^4) + 9r^2s^2(r + s)^2 =$ a ſquare.

The foregoing expreſſion might manifeſtly be made a ſquare, generally, by well known methods, but the calculation would be tedious.

It may be obſerved, that in the preceding general expreſſions for x, y, and z, any two of the quantities n, r, and s, may be taken at pleaſure, by which means the ſolution in particular caſes will be rendered much more ſimple.

Example. Take $s = 1$ and $r = -1$, then $x = n^2 - 3n - 2$, $y = -4n^2 - 3n - 1$, or, $y = 4n^2 + 3n + 1$, and $z = 5n^2 + 6n + 1$; and hence $2y^2 + 2z^2 - x^2 = 81n^4 + 174n^3 + 121n^2 + 24n =$ a ſquare.

Aſſume $9n^2 + \frac{29n}{3}$ for the root; then $81n^4 + 174n^3 + 121n^2 + 24n = (9n^2 + \frac{29n}{3})^2 = 81n^4 + 174n^3 + n^2(\frac{29}{3})^2$, which after proper reduction gives $n = \frac{-27}{31}$; whence $x = \frac{1318}{961}$, $y = \frac{1366}{961}$, and $z = -\frac{416}{961}$:

Rejecting the common denominator 961, we may take $x = 1318$, $y = 1366$, and $z = 416$: and as theſe are all even numbers they will denote the ſides of a triangle that will anſwer.

Again, put $m - 1 = n$, then $81n^4 + 174n^3 + 121n^2 + 24n = 81m^4 - 150m^3 + 85m^2 - 20m + 4 =$ a ſquare $= (9m^2 - \frac{25m}{3} + 2)^2 = 81m^4 - 150m^3 + \frac{625}{9}m^2 + 36m^2 - \frac{100}{3}m + 4$, from whence $m = \frac{15}{23}$, $n = m - 1 = -\frac{8}{23}$, $x = \frac{442}{529}$, $y = \frac{233}{529}$, and $z = \frac{255}{529}$.

Rejecting the common denominator 529, and doubling the numerators to make them even numbers, we ſhall have 884, 466, and 510 for the ſides of a triangle that will anſwer, being the ſame as were found in the firſt part of the ſolution.

From what has been done, it will be plain how other anſwers may be obtained, if any one thinks it worth the trouble of calculation.

It will be recollected that there is another ſolution to the foregoing queſtion vol. i, part ii, art. x; but beſides ſome difference in the methods of ſolution, one of the ſets of numbers here found is conſiderably ſmaller than thoſe found in the article alluded to.

ARTICLE X.

THE THEORY OF AMICABLE NUMBERS.

To the Editor of the Mathematical Repoſitory.

SIR,

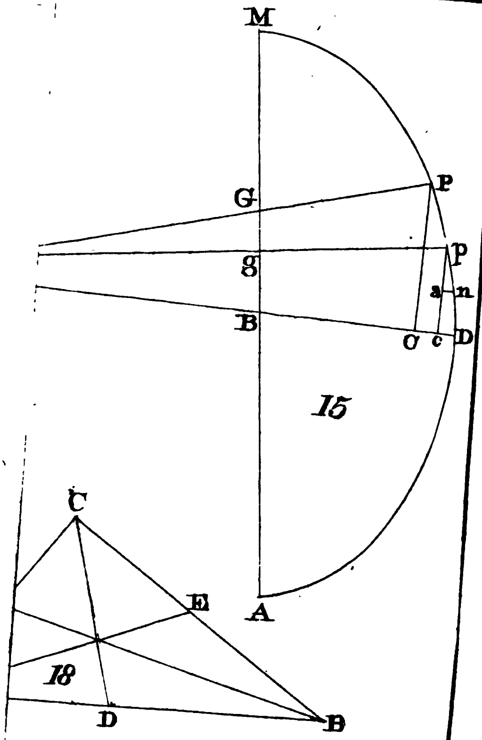
The Theory of Amicable Numbers is a ſubject which has been almoſt overlooked by the Mathematicians of this Country; and for any thing I know to the contrary, the mere Engliſh reader might have remained ignorant, both of the name and nature of this ſpecies of numbers, had not Dr. Hutton briefly explained their properties in the firſt vol. of his Mathematical Dictionary. The Article, here alluded to, is profeſſedly taken from a Tract, by F. Van Schooten, entitled " Ratio inveniendi numeros amicabiles," or the method of finding amicable numbers. It is certain that this writer borrowed his knowledge of the numbers in queſtion from Des Cartes; for he has preſerved a rule for finding them, invented by this great mathematician. As for his own enquiries they are confined to particular caſes, his intention being to diſcover a ſeries of amicable numbers increaſing progreſſively; and this appears to be the reaſon, why he gives the Carteſian rule unaccompanied by a demonſtration. I flatter myſelf that the following eſſay will be found to contain a more comprehenſive view of the ſubject, derived from conſiderations of a more general nature: at the ſame time I have not the vanity to pronounce it a complete Treatiſe. The piece compriſes ſeveral limits and general properties relating to Amicable Numbers; which may lead to a perfect Theory in abler hands. Perhaps this may have been done by ſome of the Foreign Mathematicians; but Van Schooten is the only writer on the ſubject who has come to the knowledge of

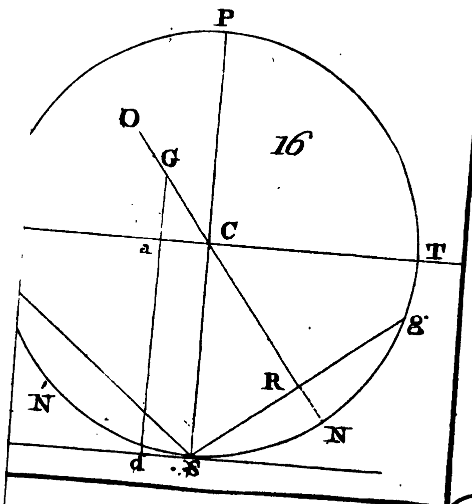
Your's, &c.

JOHN GOUGH.

Middleſhaw, near Kendal,
March 19, 1807.

Vol.II.Plate.B.Fig.15.to 18.





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Article 1. Definition. When two numbers are ſo conſtituted, that each of them is equal to the ſum of the diviſors of the other, they are called Amicable Numbers, which name was probably invented by Van Schooten.

Example. 284 and 220 are Amicable Numbers; for the ſum of the diviſors of $220 = 1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284$: alſo the diviſors of 284, taken collectively, $= 1 + 2 + 4 + 71 + 142 = 220$.

2. Let ax and ayz be two Amicable Numbers conſiſting of a common meaſure, a, multiplied by the primes x, y, and z, to find the relations of theſe primes.

Put q = the ſum of the diviſors of a; then

$$ax = a + q + (a + q)\,y + (a + q)\,z + qyz \text{ (art. 1.)};$$

for the ſame reaſon $ayz = a + q + qx$.

Invert the latter equation, and add it to the former, and

$$(a + q)(1 + x) = (a + q)(1 + y + z + yz):$$

Hence $(1 + x) = (1 + y)(1 + z)$; and $x = (y + z) + yz$.

3. Suppoſe adx, $aryz$, &c. to repreſent a pair of Amicable Numbers, conſiſting of a common meaſure a, combined with a greater number of primes than thoſe of the form in art. 2, and by art. 1, we ſhall have the two following equations,

$$(a + q)(1 + d + x) + qdx = aryz, \text{ and}$$

$$adx = (a + q)(1 + r + y + z + ry + rz + yz) + qryz:$$

Hence $(1 + d)(1 + x) = (1 + r)(1 + y)(1 + z)$.

And the ſame may be proved of any other number of primes.

Hence it appears, that if a pair of Amicable Numbers be divided by their greateſt common meaſure, and the prime diviſors of theſe quotients be ſeverally increaſed by unity, the products of the two ſets thus augmented, will be equal.

4. If ax, ayz be a pair of Amicable Numbers, we have

$$a + q + qx = ayz \text{ (art. 1.)}.$$

Hence, as $a : q :: 1 + x : yz - 1 :: (1 + y)(1 + z) : yz - 1$ (art. 2.).

Again, if adx, $aryz$, &c. be a pair of Amicable Numbers, we have $(a + q)(1 + d + x) + qdx = aryz$ (art. 3.).

Hence, as $a : q :: (1 + d)(1 + x) : ryz - (1 + d + x) :: (1 + r)(1 + y)(1 + z) : ryz - (1 + d + x)$ (art. 3.), which may alſo be proved of any other number of primes.

Hence, if a be given, q is given, but q muſt be leſs than a; and if two ſets of primes, namely, d, x, and r, y, z can be found, which will make $(1 + d)(1 + x) = (1 + r)(1 + y)(1 + z)$, &c.

and alſo give the following proportion, as $a : q :: (1+d)(1+x) : ryz - (1+d+x)$, then will adx, $aryz$ be Amicable Numbers, which is a general property. But the ſequel of this eſſay will be confined to the Carteſian form, ax, ayz.

5. Since $q+qx=ayz-a$, (art. 4.) $= qy+qz+qyz$ (art. 2.), we have $z = \frac{a+q+qy}{ay-qy-q}$, and $z+1 = \frac{a+ay}{ay-qy-q}$: for the ſame reaſon $y = \frac{a+q+qz}{az-qz-q}$, and $y+1 = \frac{a+az}{az-qz-q}$; but $x = y+z+yz$ (art. 2) $= \frac{a+q+ay+qy+ay^2}{ay-qy-q} = \frac{a+q+az+qz+az^2}{az-qz-q}$; hence if a, and either of the primes y or z be given, the other with x may be found, if the data will allow them to be primes; alſo, we have

$$x+1 = \frac{\{a(1+y)\}^2}{y-qy-q} = \frac{\{a(1+z)\}^2}{az-qz-q}.$$

6. Since $(1+x)=(1+y)(1+z)$(art. 2.) alſo $(1+y) = \frac{a+az}{az-qz-q}$, and $(1+z) = \frac{a+ay}{ay-qy-q}$ (art. 5.), we have by multiplication and ſubſtitution $(1+x) = \frac{a^2+a^2x}{(az-qz-q)(ay-qy-q)}$; therefore $(az-qz-q)(ay-qy-q) = a^2$; conſequently $(az-qz-q)$ and $(ay-qy-q)$ are ſeverally diviſors of a^2.

7. Let f be a diviſor of a^2 and put $ay-qy-q=f$; then $y = \frac{f+q}{a-q}$; but $az-qz-q = \frac{a^2}{ay-qy-q}$ (art. 6.) $= \frac{a^2}{f}$; and $z = \frac{a^2+fq}{af-fq}$; hence $y+1 = \frac{a+f}{a-q}$, and $z+1 = \frac{a^2+af}{af-fq}$; Hence, as $y+1 : z+1 :: f : a$.

8. Beſides the relations which have been ſhewn to obtain amongſt the factors of Amicable Numbers, other conditions and limits remain to be pointed out. In the firſt place then, no two of the primes x, y, and z can be equal, or in other words, x, y, and z muſt be three different numbers. For if they be not, y muſt be equal to z or x: firſt, put $y=z$, then $x=y+z+zy$, that is, $x = 2y+y^2$, or a prime = to a compoſite number, which is abſurd. Again, put $y=x$, then $x+1 = (y+1)(z+1) = (x+1)(z+1)$; hence $(z+1) = 1$, and $z=0$, which is alſo abſurd; therefore x, y, and z are different primes.

9. No one one of the three primes x, y, z can be equal to 2.

If the contrary be maintained, the abſurdity of the ſuppoſition may be proved thus: let y be leſs than x or z; then if 2 be one of the three primes, $y = 2$; but $\frac{ay + a}{ay - qy - q} = z + 1$ (art. 5.) $=$ an even number; that is, $\frac{3a}{2a - 3q}$ $=$ an even number; therefore $a =$ an even number; conſequently a may be repreſented by $2p$, $4p$, &c. $2pq$, $4pq$, &c. where p, q, &c. are primes; put $a = 2p$; then the Amicable Numbers are $2p \times x$, and $2p \times 2z = 4p \times z$, and we have (by art. 2.) $1 + x = (1 + 2)(1 + z) = 3 + 3z$: but by collecting the diviſors of each number, as in art. 1, we get $1 + 2 + p + 2p + x + 2x + px = 3 + 3p + 3x + px = 4pz$; and $1 + 2 + 4 + 2p + 4p + z + 2z + 4z + pz + 2pz = 7 + 7p + 7z + 3pz = 2px$; hence, by adding the equations, we get $7 + 7p + 7z + 7pz = 3 + 3p + 3x + 3px$, that is, $(7 + 7p)(1 + z) = (3 + 3p)(1 + x)$: but we have ſhewn that $1 + x = 3 + 3z$; conſequently $(7 + 7p)(1 + z) = (9 + 9p)(1 + z)$, an abſurdity; therefore 2 cannot be ſubſtituted for x, y, or z.

10. a is not a prime. For if a be a prime, $q = 1$, and f, in art. 7, $= a$ or 1: if $f = a$, we have $y + 1 : z + 1 :: a : a$ by that art.; that is $y = z$, which is impoſſible (art. 8.): again, let $f = 1$, and $y = \frac{f + q}{a - q}$; that is $y = 2$ or 1, but y cannot be $= 2$ (art. 9.), and 1 is not a number; hence a is not a prime.

11. In the next place it is neceſſary to find the limits of the value of q relative to a, which may be done thus: it is evident from the proportion in art. 4, that q is leſs than a; but to find a part of a which is leſs than q, proceed as follows. We have an affirmative number $z + 1 = \frac{ay + a}{ay - qy - q}$ (art. 4.), but z is odd (art. 9); conſequently $z + 1$ may be expreſſed by $2n$ an even number; hence $y = \frac{2nq + a}{2na - a - 2nq}$; but y is a whole number; therefore $2nq + a$ is greater than $2na - (a + 2nq)$: to each add $2nq - a$, and $2nq$ is greater than $na - a$; therefore $\frac{2n}{n - 1}$ will be greater than $\frac{a}{q}$; but $\frac{2n}{n - 1}$ cannot exceed 4; conſequently q is greater than $\frac{a}{4}$.

12. If a, be a power of a prime r, namely $a = r^n$, then $q = \frac{a-1}{r-1}$. For $q = 1 + r + r^2 + \ldots r^{n-1}$. Hence, if $r = 2$, $q = a - 1$, and, if $r = 3$, $q = \frac{1}{2}(a - 1)$.

13. Let $a = r^n$, r being a prime; and $q = \frac{a-1}{r-1}$ (art. 12), which is greater than $\frac{a}{4}$ (art. 11.); therefore $4a - 4$ is greater than $ra - a$: hence $(5 - r)\,a$ is greater than 4: therefore a is greater than $\frac{4}{5-r}$: but a is affirmative; conſequently the prime $r = 2$ or 3. Put $r = 3$, and $q = \frac{1}{2}(a - 2)$ (art. 11): hence $a - q = \frac{a+1}{2}$; conſequently $z + 1 = \frac{2ay + 2a}{ay + y + 1 - a}$; but z is an odd number greater than 3; therefore $z - 1 = \frac{4a - (2 + 2y)}{ay + y + 1 - a} =$ an even number $= 2m$, and $y = \frac{ma + 2a - (1 + m)}{ma + 1 + m}$, in which expreſſion the denominator is greater than half the numerator, that is, y is a fraction leſs than 2, which is abſurd; conſequently a is a power of 2, univerſally in the Amicable Numbers of Des Cartes.

14. Since $a = 2^n$, in which expreſſion n is greater than 1 (by art. 10, 13.); and f, in art. 7, is leſs than a, and a diviſor of a^2, it follows, that f is alſo a diviſor of a; conſequently we may write pf for a, and $pf - 1$ for q; hence $y = f + q = f + pf - 1$ (art. 7.), and $y + 1 = f + pf$: but as $y + 1 : z + 1 :: f : pf$; therefore $z + 1 = pf(p + 1)$, and $x + 1 = (y + 1)(z + 1) = pf^2(p + 1)^2$: now if the three values of $y + 1$, $z + 1$, and $x + 1$, ſeverally diminiſhed by unity, happen to be primes, the expreſſions $pfx = ax$, and $pfyz = ayz$ will conſtitute a pair of Amicable Numbers.

15. Put $p = 2$, then $y = 3f - 1$, $z = 6f - 1$, $x = 18f^2 - 1$, which is the ſubſtance of the rule given by Des Cartes, (vid. Franciſci a Schooten Math. Excercitationes, Leyden, pa. 423, or Hutton's Dictionary, vol. 1, pa. 105).

But there are ſtrong reaſons to believe that Amicable Numbers of the form ax, and ayz exiſt, which cannot be detected by this rule, namely, when $p = 2^n$, n being greater than unity. In theſe caſes, however, x, y, and z will be high numbers, particularly the firſt; and few perſons would be willing to undertake the trouble of determining whether they are or are not primes: I ſhall therefore conclude this eſſay with the following obſervation.

16. If the primes x, y, and z be given making $x + 1 = (y + 1)(z + 1)$, to find if they can conſtitute a pair of Amicable Numbers. Divide $z + 1$ by $y + 1$, and call the quotient p; then, if p be not 2 or a power of 2, the thing is impoſſible: but if p be ſome power of 2, divide $y + 1$ by $p + 1$, and put the quotient $= f$; then, if f be not 2 or a power thereof, the thing is impoſſible: but if $f = 2^n$, the common multiplier $a = pf$.

ARTICLE XI.

A New Solution of a Problem in Inſurance of Money on Lives: in which it is demonſtrated, that the tables for that purpoſe now in uſe are deficient by about a 67th. part of the whole.

By PHILALETHES CANTABRIGIENSIS.

1. The late increaſe of the number of offices in London for granting annuities, and inſuring ſums of money, on lives, has occaſioned my reſuming a ſubject which I had laid a ſide for many years.

Such offices, (when their number is not too great, and) when their tables are conſtructed on right principles, and their affairs are conducted with integrity, muſt prove very beneficial to the public, at the ſame time that the proprietors derive from them a reaſonable profit; but, when their calculations are erroneous, either the public muſt be injured by paying too much, or the proprietors muſt, in time, be ruined by receiving too little: and, in the latter caſe, many of the purchaſers, or their widows or children, muſt be deprived of that benefit which they expected.

On a peruſal of the works of the lateſt and moſt celebrated writers on this branch of the mathematics, my ſurpriſe has been excited at finding that many of their calculations are grounded on an erroneous principle; (viz. *taking that to be annual chance which is momentary, and allowing as much diſcount of money on the value of the chance in the firſt moment of the year as in the laſt moment of it:)* in conſequence of which the reſults are deficient by about a 67th. part of the whole. This, I doubt not, will appear to every competent judge of theſe matters, who ſhall peruſe the ſolution of the following problem. I have only to add, by way of preface, that, although I have never had any perſonal acquaintance with the writers above alluded to, and am under no ſort of obligation to them, yet I omit their names; ſuppoſing that the cauſe of truth will be ſufficiently ſerved in this inſtance by the calculations and obſervations which here follow.

PROBLEM.

2. A perſon of a given age, and in good health, is deſirous of inſuring to his executor a ſum of money denoted by S, to be paid at his death, whenever that may happen: it is required to find P, the preſent value of that ſum, according to any table of the probable duration of human life, and at any given rate of intereſt.

SOLUTION.

3. This is evidently a momentary chance; and although no tables of the probable duration of human life, that I know of, have been publiſhed for ſhorter periods than years, yet a ſolution accurate enough for common uſe may be obtained from them. It will be ſufficient alſo, in moſt caſes, to conſider the decrements of life as uniform *for the ſpace of a year*, at all ages above 7 years. Laſtly, all the accuracy requiſite in theſe matters may be obtained without the uſe of fluxions, by dividing the year into ſeconds, and uſing common algebra.

4. Theſe things being premiſed, let the number of perſons living at the given age be denoted by a, and the number living at the ages which exceed the given one by 1, 2, 3, 4, and 5 years, by b, c, d, e, and f, reſpectively. Let q denote the intereſt of £1 for one year, and R ($= 1 + q$) the amount of £1 in a year. Let m denote the number of ſeconds in a year, and r the amount of £1 in one ſecond, i. e. let $r^m = R = 1 + q$.

Now, ſince the decrements of life are ſuppoſed to be uniform during the year, and ſince the number of perſons of the given age that die in the whole of the firſt year is $a - b$, the number that will die in the mth part of that year will be $\frac{a - b}{m}$; and the chance which the perſon has to die in that part of the year, whether it be taken at the beginning, at the middle, or at the end of it, will be expreſſed by $\frac{a-b}{am}$*. The number of theſe equal chances in the whole year is evidently $= m$; and the preſent value of the ſum S, to be received on all theſe chances, is the ſum of the ſeries $\frac{a-b}{amr}S + \frac{a-b}{amr^2}S + \frac{a-b}{amr^3}S$, &c. continued to m terms, $= \frac{a-b}{am}S \times \left(\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}\text{ \&c. to }\frac{1}{r^m}\right)$ $= \frac{a-b}{am}S \times \left(\frac{1}{r-1} - \frac{1}{r^m \times (r-1)}\right)$; which expreſſion, by writing R for its equal, r^m, and reducing the fractions to a common denominator, becomes $\frac{a-b}{am}S \times \frac{R-1}{R(r-1)} = \frac{a-b}{aR}S \times \frac{R-1}{m(r-1)}$.

5. Again, ſince the number of perſons that die in the ſecond year is $b-c$, the chance of one perſon's dying during that whole year is $\frac{b-c}{a}$; and of his dying in any mth part of it will be denoted by $\frac{b-c}{am}$; and the preſent value of receiving the ſum S on m ſuch chances will be $= \frac{b-c}{amRr}S + \frac{b-c}{amRr^2}S + \frac{b-c}{amRr^3}S$, &c. to m terms, the ſum of which (by the preceding art.) is $= \frac{b-c}{amR}S \times \frac{R-1}{R(r-1)} = \frac{b-c}{aR^2}S \times \frac{R-1}{m(r-1)}$.

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* This may be illustrated, for the satisfaction of those who are not well acquainted with the doctrine of chances, as follows: According to Dr. Halley's observations, out of 427 persons living at the age of 42 years, 10 die in the space of 1 year. Therefore, by the hypothesis, 1 will die in the 10 part of a year; and the chance which a person of the aforesaid age has to live the first 10th part of a year is expressed by the fraction $\frac{426}{427}$, certainty being denoted by 1; and the chance which he has to die in the first 10th part of the year is expressed by the fraction $\frac{1}{427}$. Again, since his chance for living to the end of the first 10th part of the year is expressed by $\frac{426}{427}$; and out of 426 persons living at that age 1 will die in the next 10th part of the year, the chance which he has to live to the beginning of the second 10th part of a year and to die before the end of it, is expressed by $\frac{426}{427} \times \frac{1}{426} = \frac{1}{427}$. And, by a similar argumentation, it will appear that the chance of the persons dying in any one 10th part of the year will be expressed by the fraction $\frac{1}{427}$. And, in like manner, the chance which the same person has to die

$$\left.\begin{array}{l}\text{in 100th.}\\ \text{in 1000th.}\\ \text{in 10000th.}\end{array}\right\}\begin{array}{l}\text{part of that year will}\\ \text{be expressed by}\end{array}\left\{\begin{array}{l}\frac{1}{4270},\\ \frac{1}{42700},\\ \frac{1}{427000},\end{array}\right.$$

&c. &c.

In like manner, the preſent value of all the chances of receiving the ſum S, during the third, fourth, and fifth years, will be found to be $\frac{c-d}{aR^3}S\times\frac{R-1}{m(r-1)}$, $\frac{d-e}{aR^4}S\times\frac{R-1}{m(r-1)}$, and $\frac{e-f}{aR^5}S\times\frac{R-1}{m(r-1)}$, reſpectively. The law of continuation to the end of the table of probable duration of human life is very evident. The terms of the ſeries which thus ariſes are as follows:

For the 1ſt. year $\frac{a-b}{aR}S\times\frac{R-1}{m(r-1)}$,

2d. .... $\frac{b-c}{aR^2}S\times\frac{R-1}{m(r-1)}$,

3d. .... $\frac{c-d}{aR^3}S\times\frac{R-1}{m(r-1)}$,

4th. .... $\frac{d-e}{aR^4}S\times\frac{R-1}{m(r-1)}$,

5th. .... $\frac{e-f}{aR^5}S\times\frac{R-1}{m(r-1)}$, &c. &c. to the end of the table; i. e. till one of the fractional factors becomes $= 0$.

6. Now, ſince the factor $\frac{R-1}{m(r-1)}$ (which, for the ſake of brevity, I denote by F,) is common to all the terms, as well as the factor S, if we can find any eaſy way of computing the ſum of the finite ſeries

$$\frac{a-b}{aR}+\frac{b-c}{a}+\frac{c-d}{aR^3}+\frac{d-e}{aR^4}+\frac{e-f}{aR^5}, \text{ \&c.}$$

we ſhall readily obtain the preſent value of all the chances which the executor has to receive the ſum S. But the ſum of this ſeries is evidently equal to the difference of the ſums of theſe two ſeries,

$$\left.\begin{array}{ll} \textit{viz.} & \frac{a}{aR}+\frac{b}{aR^2}+\frac{c}{aR^3}+\frac{d}{aR^4}+\frac{e}{aR^5}, \text{ \&c.} \\ \text{and} & \frac{b}{aR}+\frac{c}{aR^2}+\frac{d}{aR^3}+\frac{e}{aR^4}+\frac{f}{aR^5} \text{ \&c.} \end{array}\right\} \begin{array}{l}\text{till the nu-}\\ \text{merator be-}\\ \text{comes} = 0.\end{array}$$

The firſt of theſe ſeries is manifeſtly $=$

$$\frac{1}{R}\times\left(1+\frac{b}{aR}+\frac{c}{aR^2}+\frac{d}{aR^3}+\frac{e}{aR^4}, \text{ \&c.}\right)=\frac{1}{R}\times$$

$(1+\mathcal{A})$

$(1+A)$, the letter A being put for the value of an annuity of £1 on the life of the perſon propoſed in the queſtion. And ſince the ſecond ſeries is obviouſly $= A$, we have their difference $= \frac{1+A}{R} - A$, and thence P, the value ſought, $= FS \times \left(\frac{1+A}{R} - A\right) = FS \times \frac{1-(R-1)A}{R} = FS \times \frac{1-qA}{R}$.

7. Here it is worthy of remark, that the theorem now in general uſe is $P = S \times \left(\frac{1+A}{R} - A\right)$. Whatever ratio therefore, F ſhall be found to bear to 1, ſuch will be the ratio of the true value of the inſurance to that which is given by the rule now in uſe.

8. But the value of $F\left(= \frac{R-1}{m(r-1)}\right)$ is eaſily obtained thus: ſince $r^m = R = 1+q$, we have $r = R^{\frac{1}{m}} = (1+q)^{\frac{1}{m}} = 1 + \frac{1}{m}q + \frac{1-m}{m.2m}q^2 + \frac{1-m}{m.2m} \times \frac{1-2m}{3m}q^3 + \frac{1-m}{m.2m} \times \frac{1-2m}{3m} \times \frac{1-3m}{4m}q^4$, &c. and $m(r-1) = q + \frac{1-m}{2m}q^2 + \frac{1-m}{2m} \times \frac{1-2m}{3m}q^3 + \frac{1-m}{2m} \times \frac{1-2m}{3m} \times \frac{1-3m}{4m}q^4$, &c. which expreſſion, as m is greater than 31000000, is ſo nearly $= q - \frac{1}{2}q^2 + \frac{1}{3}q^3 - \frac{1}{4}q^4$, &c. that the difference, in this caſe, is inconſiderable. And this ſeries is known to be $=$ the hyperbolic logarithm* of $1+q$, or R, which logarithm may be denoted by L. We therefore now have $F = \frac{R-1}{L}$, by which expreſſion, and a table of hyperbolic logarithms, its numerical value may quickly be obtained, when the value of R is given. Its value may eaſily be found alſo by ſeries, as follows: $F = \frac{R-1}{L} = q \div (q - \frac{1}{2}q^2 + \frac{1}{3}q^3 - \frac{1}{4}q^4$, &c.$)$ $= 1 + \frac{1}{2}q - \frac{1}{12}q^2 + \frac{1}{24}q^3$, &c. which ſeries, as q is a ſmall quantity, is evidently greater than 1. It will indeed be nearly equal to, but always ſomewhat greater than, $\sqrt{1+q} = \sqrt{R}$.

9. The general ſolution of the problem being now completed, an example or two, by way of illuſtration, may be proper.

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* See the *Scriptores Logarithmici*, vol. i. pa. 244 and 245.

Example 1. Let it be required to find the numerical value of F, when $R = 1{\cdot}03$, which is the common rate of intereſt allowed at the offices of inſurance on lives,

Here $R - 1 = 0{\cdot}03$; and the hyp. log. of $R = 0{\cdot}029558 = L$; and $(R - 1) \div L$ is $= 1{\cdot}014926$, the value of F required.

The ſame value of F may be found by the ſeries given for that purpoſe in the preceding article, without the uſe of hyperbolic logarithms, thus:

$$\begin{array}{rl} 1 + \frac{1}{2}q & = 1{\cdot}0150000, \\ + \frac{1}{24}q^3 & = \underline{0{\cdot}0000011,} \\ \text{The ſum} & = 1{\cdot}0150011, \\ - \frac{1}{12}qq & = \underline{-0{\cdot}0000750,} \\ \text{The difference} & = \underline{1{\cdot}0149261,}\ \text{the value of } F \text{ required.} \end{array}$$

If we now divide 1 by 0·014926, the quotient will be 66·997, or 67 very nearly; which ſhows that F exceeds 1 by ſomewhat more than a 67th part of itſelf. The loſs to the offices, therefore, by the erroneous rule now in uſe, is ſomewhat more than £1 upon every £67 which they receive: it is indeed full £149 5*s*. upon every £10000.

If the rate of intereſt allowed were greater than £3 *per cent.* the error of the rule would alſo be greater than that which has been now computed.

Example 2. Let it be required to find the preſent value of £1000 to be received at the death of a perſon now 50 years of age, and in good health, taking the probable duration of human life to be ſuch as is ſhown by the Northampton table, and allowing 3 *per cent.* intereſt of money.

Here $A = 12{\cdot}436$, $R = 1{\cdot}03$, and $S = 1000$; from which we have $qA = 0{\cdot}37308$, $1 - qA = 0{\cdot}62692$, $(1 - qA) \div R = 0{\cdot}60866$, and $S \times (1 - qA) \div R =$ £608·66, the anſwer required, by the rule now in uſe.

The value of the annuity here uſed, *viz.* 12·436, is taken from Dr. Price's *Obſervations on Reverſionary Payments*, Edit. 6th, vol. ii. p. 315, where the numbers are continued to no more than three places of decimals, in conſequence of which the laſt figure in the above calculation could not be depended upon, if it were not found by a different computation by the ſame erroneous rule. Now if we multiply the number 608·66 by 1·014926, we ſhall have the true anſwer (*viz.* $FS \times (1 - qA) \div R$,) = £617·744, which exceeds the erroneous anſwer by £9·084, or £9 1*s*. 8*d*.

REMARKS.

REMARKS.

1. This error deſerves the attention of the directors of offices for inſuring ſums of money on lives. It is indeed ſaid, that the proprietors of an office, in which a table of rates, grounded on the Northampton table and computed by the aforeſaid erroneous theorem, has been uſed for the ſpace of twenty years, have made conſiderable gain by it. This may be true: and it may be accounted for by obſerving, that the expectation of human life is, in general, ſomewhat longer than is ſhown by the Northampton table; and that the intereſt of money, during that period, has been more than 3 *per cent*, which is the rate allowed to thoſe who make inſurances at that office. *But ſtill their gain is not ſo great as they imagine.*

The preſent competition of offices of this kind, for the favour of the public, may poſſibly induce ſome of them to offer lower rates of inſurance, grounded on a table which ſhows the duration and gradual waſte of human life to be ſuch as it generally is in this Iſland; and ſhould ſuch tables be conſtructed by the theorem which has long been in uſe, that is, on the erroneous principle which I have now pointed out, the conſequence muſt, in time, be ruinous to the proprietors, and to a multitude of widows and orphans, who muſt be deprived of that ſupport, in the hour of diſtreſs, which was purchaſed for them by their huſbands and fathers. Thoſe proprietors therefore, who pay a due attention to juſtice, and to their own permanent intereſt, will not lower the preſent rates of inſurance, without conſulting ſome very able mathematician, who alſo has had much experience in calculations of this kind.

2. I muſt not omit to mention, that the ſame erroneous principle, which is above pointed out, has been uſed in conſtructing the tables of rates for inſurance of ſums of money when more than one life is concerned; in which caſe the effect of it varies with the number of lives.

3. I ought alſo to add, that I have neither private intereſt to ſerve, nor perſonal enmity to gratify, on this occaſion. The courſe of my ſtudies led me to an acquaintance with the doctrine of annuities and inſurance on lives in my youth; and the many applications which have been made to me, of late years, by friends and acquaintance, for my opinion of the valuation of inſurances on lives, have induced me to examine the principles on which the tables that have long been uſed for that purpoſe are conſtructed; and the examination has led me to the diſcovery of the error which is above demonſtrated; a diſcovery which I take to be of no ſmall importance to many individuals, if not to the public at large.

PHILALETHES CANTABRIGIENSIS.

ARTICLE XII.

An inveſtigation of Theorems for finding the ſums of certain Infinite Series, &c. By Mr. JAMES CUNLIFFE.

It has been ſhewn at art. 5, page 16, that

$$\frac{s}{1-2cx+x^2} = \text{ſin.}\,z+x\,\text{ſin.}\,2z+x^2\,\text{ſin.}\,3z+x^3\,\text{ſin.}\,4z+\&c.$$

where s and c denote the ſine and coſine of the circular arc z, radius 1,

1. Multiplying the preceding expreſſion by $\dot{x}$, and taking the fluents, $\int\frac{s\dot{x}}{1-2cx+x^2} = R = x\,\text{ſin.}\,z+\frac{1}{2}x^2$ ſin. $2z+\frac{1}{3}x^3\,\text{ſin.}3z+\frac{1}{4}x^4\,\text{ſin.}\,4z+\&c.$ where R denotes the length of a circular arc, radius 1 and tangent $\frac{sx}{1-cx}$, or ſine $\frac{sx}{\sqrt{(1-2cx+x^2)}}$.

2. In the place of c in the laſt expreſſion write $-c$, and it becomes $\int\frac{s\dot{x}}{1+2cx+x^2}=S=x\text{ſin.}z-\frac{1}{2}x^2\text{ſin.}2z+\frac{1}{3}x^3\text{ſin.}3z-\frac{1}{4}x^4\text{ſin.}4z$ $+\&c.$ where S denotes the length of a circular arc, radius 1 and tangent $\frac{sx}{1+cx}$, or ſine $\frac{sx}{\sqrt{(1+2cx+x^2)}}$.

3. Half the ſum of the expreſſions 1 and 2 is $\frac{1}{2}(R+S)=x\text{ſin.}z+\frac{1}{3}x^3\text{ſin.}3z+\frac{1}{5}x^5\text{ſin.}\,5z+\frac{1}{7}x^7\text{ſin.}\,7z+\&c.$ where R and S are as before ſtated; or R + S denotes the length of a circular arc, radius 1 and tangent $\frac{2sx}{1-x^2}$, or ſine $\frac{2sx}{\sqrt{\{1+2x^2\times(1-2c^2)+x^4\}}} = \frac{2sx}{\sqrt{\{1+2x^2\times(2s^2-1)+x^4\}}}$.

4. Again half the difference of the expreſſions 1 and 2 is $\frac{1}{2}(R-S)=\frac{1}{2}x^2\,\text{ſin.}\,2z+\frac{1}{4}x^4\,\text{ſin.}\,4z+\frac{1}{6}x^6\,\text{ſin.}\,6z+\&c.$ where R − S denotes the length of a circular arc, radius 1

and tangent $\frac{2scx^2}{1 + x^2 \times (s^2 - c^2)}$, or ſine

$$\frac{2scx^2}{\sqrt{\left\{1 + 2x^2 \times (1 - 2c^2) + x^4\right\}}} = \frac{2scx^2}{\sqrt{\left\{1 + 2x^2 \times (2s^2 - 1) + x^4\right\}}}.$$

Examples to the formula No. 3.

Example 1. Let z denote an arc of $45°$: then ſin. $z = \sqrt{\tfrac{1}{2}}$; ſin. $3z = \sqrt{\tfrac{1}{2}}$; ſin. $5z = -\sqrt{\tfrac{1}{2}}$; ſin. $7z = -\sqrt{\tfrac{1}{2}}$ &c. which being written in the formula No. 3, give $\tfrac{1}{2}(R + S) = \sqrt{\tfrac{1}{2}} \times (x + \tfrac{1}{3}x^3 - \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \tfrac{1}{9}x^9 + \tfrac{1}{11}x^{11} - \tfrac{1}{13}x^{13} - \tfrac{1}{15}x^{15}$ + &c.) and dividing by $\sqrt{\tfrac{1}{2}}$, $\tfrac{1}{2}(R + S)\sqrt{2} = x + \tfrac{1}{3}x^3 - \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \tfrac{1}{9}x^9 + \tfrac{1}{11}x^{11}$ — &c. where R + S denotes the length of a circular arc, radius 1 and tangent $\frac{x\sqrt{2}}{1 - x^2}$, or ſine $\frac{x\sqrt{2}}{\sqrt{(1 + x^4)}}$.

Now the ſeries $x + \tfrac{1}{3}x^3 - \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7$ + &c. is manifeſtly $= \int \frac{\dot{x}}{1 + x^4} + \int \frac{x^2\dot{x}}{1 + x^4}$; therefore the ſum of theſe two fluents is equal to the length of a circular arc, radius 1 and tangent $\frac{x\sqrt{2}}{1 - x^2}$, or ſine $\frac{x\sqrt{2}}{\sqrt{(1 + x^4)}}$.

Take $x = 1$, and then $\tfrac{1}{2}(R + S)\sqrt{2} = 1 + \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{13} - \tfrac{1}{15}$ + &c. $= \frac{4}{1 \cdot 3} - \frac{12}{5 \cdot 7} + \frac{20}{9 \cdot 11} - \frac{28}{13 \cdot 15}$ + &c. where R + S in the preſent caſe denotes the circular arc, radius 1 and ſine 1, or the quadrantal arc.

Example 2. Let z denote an arc of $60°$; then ſin. $z = \tfrac{1}{2}\sqrt{3}$; ſin. $3z = 0$; ſin. $5z = -\tfrac{1}{2}\sqrt{3}$; ſin. $7z = \tfrac{1}{2}\sqrt{3}$; ſin. $9z = 0$; ſin. $11z = -\tfrac{1}{2}\sqrt{3}$; ſin. $13z = \tfrac{1}{2}\sqrt{3}$; ſin. $15z = -\tfrac{1}{2}\sqrt{3}$, &c. and theſe values being written in the formula No. 3, give $\tfrac{1}{2}(R + S) = \tfrac{1}{2}\sqrt{3} \times (x - \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 - \tfrac{1}{11}x^{11} + \tfrac{1}{13}x^{13} - \tfrac{1}{17}x^{17}$ + &c.) and dividing by $\tfrac{1}{2}\sqrt{3}$, $\frac{1}{\sqrt{3}} \times (R + S) = x - \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 - \tfrac{1}{11}x^{11} + \tfrac{1}{13}x^{13} - \tfrac{1}{17}x^{17}$ + &c.

Now the ſeries $x - \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 - \tfrac{1}{11}x^{11} + \tfrac{1}{13}x^{13} - \tfrac{1}{17}x^{17}$ + &c. $= \int \frac{\dot{x}}{1 - x^6} - \int \frac{x^4\dot{x}}{1 - x^6}$; therefore the difference of the fluents $\int\frac{\dot{x}}{1-x^6}$ and $\int\frac{x^4\dot{x}}{1-x^6}$ is equal to $\frac{1}{\sqrt{3}}\times(R+S)$ where R + S denotes the length of a circular arc, radius 1 and tangent $\frac{x\sqrt{3}}{1-x^2}$, or ſine $\frac{x\sqrt{3}}{\sqrt{(1+x^2+x^4)}}$.

Take $x=1$; then $\frac{1}{\sqrt{3}}\times(R+S)=1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\frac{1}{17}$ + &c. $=\frac{4}{1\cdot5}-\frac{4}{7\cdot11}+\frac{4}{13\cdot17}-\frac{4}{19\cdot23}$ + &c. and dividing by 4, $\frac{1}{4\sqrt{3}}\times(R+S)=\frac{1}{1\cdot5}-\frac{1}{7\cdot11}+\frac{1}{13\cdot17}-\frac{1}{19\cdot23}$ + &c. in which circumſtance R + S denotes the quadrantal arc of a circle radius 1.

Example 3. Let z denote an arc of 30°; then ſin. $z=\frac{1}{2}$; ſin. $3z=1$; ſin. $5z=\frac{1}{2}$; ſin. $7z=-\frac{1}{2}$; ſin. $9z=-1$; ſin. $11z=-\frac{1}{2}$ &c. and theſe being written in the formula No. 3, give $\frac{1}{2}(R+S)=\frac{1}{2}\times(x+\frac{2}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7-\frac{2}{9}x^9-\frac{1}{11}x^{11}+\&c.)$, or $R+S=x+\frac{2}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7-\frac{2}{9}x^9-\frac{1}{11}x^{11}+\&c.$ where R + S denotes the length of a circular arc, radius 1 and tangent $\frac{x^2}{1-x^2}$.

Now the ſeries $x+\frac{2}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7-\frac{2}{9}x^9-\frac{1}{11}x^{11}+\&c.$ is pretty evidently $=\int\frac{\dot{x}}{1+x^6}+\int\frac{2x^2\dot{x}}{1+x^6}+\int\frac{x^4\dot{x}}{1+x^6}$; therefore the ſum of theſe three fluents is equal to the length of a circular arc, radius 1 and tangent $\frac{x}{1-x^2}$.

Examples to the formula No. 4.

Example 1. Let z denote an arc of 45°; then ſin. $2z=1$; ſin. $4z=0$; ſin. $6z=-1$; ſin. $8z=0$; ſin. $10z=1$; ſin. $12z=0$; ſin. $14z=-1$ &c. and theſe being written in the formula No. 4, give $\frac{1}{2}(R-S)=\frac{1}{2}x^2-\frac{1}{6}x^6+\frac{1}{10}x^{10}-\frac{1}{14}x^{14}+\&c.$ or $R-S=x^2-\frac{1}{3}x^6+\frac{1}{5}x^{10}-\frac{1}{7}x^{14}+\&c.$ where R — S denotes the length of a circular arc, radius 1 and tangent x^2, and this is alſo known from more ſimple principles.

Example 2. Let z denote an arc of 60°: then $\sin 2z = \frac{1}{2}\sqrt{3}$; $\sin 4z = -\frac{1}{2}\sqrt{3}$; $\sin 6z = 0$; $\sin 8z = \frac{1}{2}\sqrt{3}$: $\sin 10z = -\frac{1}{2}\sqrt{3}$; $\sin 12z = 0$ &c. and these being written in the formula No. 4, give $\frac{1}{2}(R - S) = \frac{1}{2}\sqrt{3} \times (\frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{8}x^8 - \frac{1}{10}x^{10} + \&c.)$, and dividing by $\frac{1}{2}\sqrt{3}$ gives $\frac{1}{\sqrt{3}} \times (R-S) = \frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{8}x^8 - \frac{1}{10}x^{10} + \&c.$ where $R - S$ denotes the length of a circular arc radius 1, and tangent $\frac{\frac{1}{2}x^2\sqrt{3}}{1 + \frac{1}{2}x^2} = \frac{x^2\sqrt{3}}{2 + x^2}$. Now $\frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{8}x^8 - \frac{1}{10}x^{10} + \&c.$ is pretty manifestly equal to $\int \frac{x\dot{x}}{1 - x^6} - \int \frac{x^3\dot{x}}{1 - x^6}$; therefore, $\frac{1}{\sqrt{3}} \times$ into the circular arc radius 1, and tangent $\frac{x^2\sqrt{3}}{2 + x^2}$ is equal to the difference of the fluents $\int \frac{x\dot{x}}{1 - x^6}$ and $\int \frac{x^3\dot{x}}{1 - x^6}$.

Take $x = 1$, then $\frac{1}{\sqrt{3}} \times (R-S) = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{10} + \frac{1}{14} - \frac{1}{16} + \&c.$ $= \frac{2}{2.4} + \frac{2}{8.10} + \frac{2}{14.16}$ &c. and multiplying by 2, $\frac{2}{\sqrt{3}} \times (R-S) = \frac{1}{1.2} + \frac{1}{4.5} + \frac{1}{7.8}$ &c. in which circumstance $R - S$ denotes a circular arc radius 1, and tangent $\frac{1}{\sqrt{3}}$: or $R - S$ in the present case is an arc of 30°.

Example 3. Let z denote an arc of 30°; then $\sin 2z = \frac{1}{2}\sqrt{3}$; $\sin 4z = \frac{1}{2}\sqrt{3}$; $\sin 6z = 0$; $\sin 8z = -\frac{1}{2}\sqrt{3}$; $\sin 10z = -\frac{1}{2}\sqrt{3}$; $\sin 12z = 0$, &c. and these being written in the formula No. 4, give $\frac{1}{2}(R - S) = \frac{1}{2}\sqrt{3} \times (\frac{1}{2}x^2 + \frac{1}{4}x^4 - \frac{1}{8}x^8 - \frac{1}{10}x^{10} + \&c.)$, and dividing by $\frac{1}{2}\sqrt{3}$, $\frac{1}{\sqrt{3}} \times (R - S) = \frac{1}{2}x^2 + \frac{1}{4}x^4 - \frac{1}{8}x^8 - \frac{1}{10}x^{10} + \&c.$ where $R - S$ denotes the length of a circular arc, radius 1, and tangent $\frac{\frac{1}{2}x^2\sqrt{3}}{1 - \frac{1}{2}x^2} = \frac{x^2\sqrt{3}}{2 - x^2}$.

Now $\frac{1}{2}x^2 + \frac{1}{4}x^4 - \frac{1}{8}x^8 - \frac{1}{10}x^{10} + \&c. = \int \frac{x\dot{x}}{1 + x^6} + \int \frac{x^3\dot{x}}{1 + x^6}$; therefore the sum of the fluents $\int \frac{x\dot{x}}{1 + x^6} + \int \frac{x^3\dot{x}}{1 + x^6}$, is equal to $\frac{1}{3\sqrt{}}$ × circular arc radius 1, and tangent $\frac{x^2\sqrt{3}}{2-x^2}$.

Take $x = 1$, then $\frac{1}{\sqrt{3}} \times (R - S) = \frac{1}{2} + \frac{1}{4} - \frac{1}{8} - \frac{1}{10} + \frac{1}{14} + \frac{1}{16} - \frac{1}{20} - \frac{1}{22} + \&c. = \frac{6}{2.4} - \frac{18}{8.10} + \frac{30}{14.16} - \frac{42}{20.22} + \&c.$ and multiplying by $\frac{2}{3}$, $\frac{2}{3\sqrt{3}} \times (R - S) = \frac{4}{2.4} - \frac{12}{8.10} + \frac{20}{14.16} - \frac{28}{20.22} + \&c. = \frac{1}{1.2} - \frac{3}{4.5} + \frac{5}{7.8} - \frac{7}{10.11} + \&c.$ where $R - S$ denotes a circular arc, radius 1, and tangent $\sqrt{3}$, or an arc of 60°.

ARTICLE XIII.

On the attraction of an infinite solid elliptic cylinder. By Mr. Thomas Knight, of Papcastle, near Cockermouth.

To the Editor of the Mathematical Repository.

SIR,

I send the following Problem for your Repository, because it contains a fluent, the finding of which is somewhat laborious, and as it will introduce some remarks I have to make on a passage of the Mécanique Céleste of La Place.

LEMMA. Let $\dot{M}$ be the element of a solid body, f its distance from an attracted point, then, the force being inversely as the square of the distance, $\dot{M} \div f^2$ is the attraction of that element on the point. By resolving this, we shall easily see that the attraction of the point by $\dot{M}$ in the direction of any other line u is $\frac{\dot{M}}{f^2} \times \left(\frac{\dot{f}}{\dot{u}}\right)$; whence the attraction of the whole solid in the direction u is $\int \frac{\dot{M}}{f^2} \times \left(\frac{\dot{f}}{\dot{u}}\right) =$ (if we put $\int \frac{\dot{M}}{f} = V$)

$$-\left(\frac{\dot{V}}{\dot{u}}\right).$$

PROBLEM

PROBLEM.

Let XX, (fig. 19, pl. C,) be an infinite solid elliptic cylinder, p a point lying in the prolongation of the transverse diameter of the generating ellipsis. It is required to find the attraction the solid exercises on the point?

Let rq be any line in the solid, parallel to its axis; r the point where it meets a plane perpendicular to the axis, and passing through p; mr a line perpendicular to Ap, or to the transverse axis of the generating ellipsis, A being its centre.

Put $p\mathrm{A} = u$, $rq = t$, $\mathrm{A}m = x$, $mr = y$, $pq = f = \surd\left\{(u-x)^2+y^2+t^2\right\}$. Suppose p to be attracted by an element placed at q, we have, (because $\dot{\mathrm{M}} = \dot{t}\dot{y}\dot{x}$) $\int\frac{\dot{\mathrm{M}}}{f} = \mathrm{V} =$

$$\iiint\frac{\dot{t}\dot{y}\dot{x}}{\surd\left\{(u-x)^2+y^2+t^2\right\}}$$

: whence the attraction of the solid in the direction u, on the particle $= -\left(\frac{\dot{\mathrm{V}}}{\dot{u}}\right)$, by the lemma,

$$= \iiint\frac{\dot{t}\dot{y}\dot{x}\,(u-x)}{\left\{(u-x)^2+y^2+t^2\right\}^{\frac{3}{2}}}.$$

Let the equation of the generating ellipsis be $\tau^2 y^2 = k^2 - x^2$. The fluent relative to t must be taken from $t = -\infty$ to $t = \infty$: the fluent relative to y from $y = \frac{-\surd(k^2-x^2)}{\tau}$ to $y = \frac{\surd(k^2-x^2)}{\tau}$; and lastly that relative to x from $x = -k$ to $x = k$. We have then successively,

$$-\left(\frac{\dot{\mathrm{V}}}{\dot{u}}\right) = \iiint\frac{\dot{t}\dot{y}\dot{x}\,(u-x)}{\left\{(u-x)^2+y^2+t^2\right\}^{\frac{3}{2}}} = 2\iint\frac{\dot{y}\dot{x}\,(u-x)}{(u-x)^2+y^2} = 4\int\dot{x}\times\text{ang. tang.}\frac{\surd(k^2-x^2)}{\tau\,(u-x)}.$$

The last fluent is the only one of any difficulty. Because $\int u\dot{v} = uv - \int v\dot{u}$, we have first $\int\dot{x}\times\text{ang. tang.}\frac{\surd(k^2-x^2)}{\tau\,(u-x)} =$

$$x\times\text{ang. tang.}\frac{\surd(k^2-x^2)}{\tau\,(u-x)} - \int x\,\frac{\left(\frac{\surd(k^2-x^2)}{\tau\,(u-x)}\right)^{\bullet}}{1+\left(\frac{\surd(k^2-x^2)}{\tau\,(u-x)}\right)^2},$$

the former part of which vanishes when taken from $x = -k$ to $x = k$. There remains then to find

$$\int x\frac{\left(\frac{\sqrt{(k^2-x^2)}}{\tau(u-x)}\right)^{\bullet}}{1+\left(\frac{\sqrt{(k^2-x^2)}}{\tau(u-x)}\right)^2}=\int\left\{\begin{array}{l}+\ \frac{\tau x\dot{x}\sqrt{(k^2-x^2)}}{\tau^2(u-x)^2+k^2-x^2}\\ -\ \frac{\tau(u-x)x^2\dot{x}}{(\tau^2(u-x)^2+k^2-x^2)\sqrt{(k^2-x^2)}}.\end{array}\right.$$

In order to make this rational, put $\frac{-(k+x)}{x-k}=z^2$, whence $x=\frac{kz^2-k}{z^2+1}$, $\sqrt{(k^2-x^2)}=\frac{-2kz}{z^2+1}$, $\dot{x}=\frac{4kz\dot{z}}{(z^2+1)^2}$. By substituting these values and making proper reductions, we find

$$\int x\frac{\left(\frac{\sqrt{(k^2-x^2)}}{\tau(u-x)}\right)^{\bullet}}{1+\left(\frac{\sqrt{(k^2-x^2)}}{\tau(u-x)}\right)^2}=\int 2d\times\frac{z^4-z^2-az^2+a}{(z^2+1)(z^2+g)(z^2+f)}\times\dot{z},$$

where $g=\frac{b}{2}-\sqrt{(\frac{b^2}{4}-a^2)}$, $f=\frac{b}{2}+\sqrt{(\frac{b^2}{4}-a^2)}$, $a=\frac{u+k}{u-k}$, $d=\frac{k^2}{\tau(u-k)}$, $b=\frac{2(u+k)}{u-k}+\frac{4k^2}{\tau^2(u-k)^2}$; it is plain that $\frac{b^2}{4}-a^2$ is positive, as are also f and g. By resolving the above fraction, we have

$$\int x\frac{\left(\frac{\sqrt{(k^2-x^2)}}{\tau(u-x)}\right)^{\bullet}}{1+\left(\frac{\sqrt{(k^2-x^2)}}{\tau(u-x)}\right)^2}=\int 2d\left\{\begin{array}{l}\frac{(2+2a)\dot{z}}{(f-1)(g-1)(z^2+1)}+\frac{(g+1)(g+a)\dot{z}}{(1-g)(f-g)(z^2+g)}\\ +\ \frac{(f+1)(f+a)\dot{z}}{(1-f)(g-f)(z^2+f)}\end{array}\right.$$

whose fluent

$$=2d\left\{\begin{array}{l}\frac{2+2a}{(f-1)(g-1)}\times\text{Arc. tan}=z+\frac{(g+1)(g+a)}{(1-g)(f-g)\sqrt{g}}\times\text{Arc. ta}=\frac{z}{\sqrt{g}}\\ +\ \frac{(f+1)(f+a)}{(1-f)(g-f)\sqrt{f}}\times\text{Arc. tan}=\frac{z}{\sqrt{f}},\end{array}\right.$$

which taken from $x=-k$, to $x=k$, gives

$$2d\left\{\frac{2+2a}{(f-1)(g-1)}+\frac{(g+1)(g+a)}{(1-g)(f-g)\sqrt{g}}+\frac{(f+1)(f+a)}{(1-f)(g-f)\sqrt{f}}\right\}\frac{\pi}{2},$$

π being half the circumference when the radius is unity. To reduce this to a simpler form, we have

$$f=\frac{u+k}{u-k}+\frac{2k^2}{\tau^2(u-k)^2}+\frac{2k}{\tau(u-k)^2}\sqrt{(u^2-\frac{k^2(\tau^2-1)}{\tau^2})},$$

$$g=\frac{u+k}{u-k}+\frac{2k^2}{\tau^2(u-k)^2}-\frac{2k}{\tau(u-k)^2}\sqrt{(u^2-\frac{k^2(\tau^2-1)}{\tau^2})},$$

$$fg = \frac{(u+k)^2}{(u-k)^2},\; f-g = \frac{4k}{\tau(u-k)^2}\sqrt{\left(u^2 - \frac{k^2(\tau^2-1)}{\tau^2}\right)};$$

$$\sqrt{f} = \frac{k+\tau\sqrt{\left(u^2-\frac{k^2(\tau^2-1)}{\tau^2}\right)}}{\tau(u-k)},\; \sqrt{g} = \frac{-k+\tau\sqrt{\left(u^2-\frac{k^2(\tau^2-1)}{\tau^2}\right)}}{\tau(u-k)}$$

and by substituting these values, we find at last, the attraction of the solid

$$= -\left(\frac{\dot{V}}{\dot{u}}\right) = \frac{4\pi\tau}{\tau^2-1}\left\{u - \sqrt{\left(u^2 - \frac{k^2(\tau^2-1)}{\tau^2}\right)}\right\}.$$

SCHOLIUM. In the investigation by Mr. La Place, of the figure of Saturn'sRing, Méc.Cél. Liv. 3, Chap. 6, it appears to me, that his conclusions are rendered invalid by an error which I shall point out. He has rightly shewn, that the partial differential equation $\left(\frac{ddV}{du^2}\right) + \left(\frac{ddV}{dz^2}\right) = 0$, and consequently, its integral $V = \varphi(u + z\sqrt{-1}) + \psi(u - z\sqrt{-1})$, which belong to an infinite cylinder, are also nearly true relative to the ring, if u and z are very small with respect to a, (see his work). But it by no means follows, that the *form* of the function in the one case, has the least similitude, or its value any approximation to that in the other. Yet $\int \frac{dt}{\sqrt{\{(u-x)^2+y^2+t^2\}}}$ belongs accurately to an infinite cylinder, notwithstanding the circuitous manner in which he arrives at it, and by which it would appear that he does not mean it to belong to such a cylinder.

Relatively to a ring, whose radius a is infinite, and whose dimensions and likewise the distance u are small, with respect to a, he would have had the integral

$$\int \frac{(a+x)\,d\bar{\omega}}{\sqrt{\{2a^2 + 2au + 2ax - (2a^2 + 2au)\cos\bar{\omega}\}}},$$

from which to find the form of the function $\varphi(u)$.

Let *rs*, (fig. 20, pl. C,) be a circular line, the breadth evanescent, *mn* a touching line in the same plane; surely a particle at *d*, will not be equally attracted by each of these lines. If that be the case, remove it to *e*, and the circumference will not attract it all, which is plainly absurd, and yet the evident conclusion from the author's reasoning.

ARTICLE XIV.

Two Indeterminate Problems. By Mr. Cunliffe.

PROBLEM I.

To find two scalene triangles, such, that the lengths of the perpendiculars from the angles upon the opposite sides, and segments of the sides made by the said perpendiculars, may be expressed by rational numbers; and also, that the perimeters and areas of the two triangles may be equal, each to each respectively.

SOLUTION.

Let ACB and A′C′B′, (fig. 21, 22, pl. C,) represent two triangles of the kind mentioned in the question. In order that the perpendiculars and segments of the sides made by the perpendiculars may be expressed by rational numbers, put AC $= m^2+n^2$, BC $= p^2+q^2$, AD $= m^2-n^2$, BD $= p^2-q^2$; then CD $= 2mn = 2pq$: also, put A′C′ $= m'^2+n'^2$, B′C′ $= p'^2+q'^2$, A′D′ $= m'^2-n'^2$, B′D′ $= p'^2-q'^2$, then C′D′ $= 2m'n' = 2p'q'$.

Now by the question, the perimeters and areas of the two triangles are to be equal each to each, that is, the perimeter of the one triangle is to be equal to the perimeter of the other; and the areas are in like manner to be equal. But the area of a triangle is known to be equal to the rectangle under the perimeter and half the radius of the inscribed circle, therefore the radii of the inscribed circles of the two triangles, must be equal to each other. These things being noticed, we shall proceed with the notation of the triangle ACB, to obtain expressions for the perimeter, and the radius of the inscribed circle.

The perimeter of the triangle ACB, is $2(m^2+p^2)$, and its area $\frac{1}{2}\text{CD} \times \text{AB} = mn(m^2-n^2+p^2-q^2) = mn(m^2-n^2+p^2-m^2n^2 \div p^2) = (mn \div p^2)(p^2m^2-p^2n^2+p^4-m^2n^2)$, because $pq = mn$, or $q = mn \div p$; and dividing the area by half the perimeter, gives $(mn \div p^2)(p^2-n^2)$, for the radius of the inscribed circle. And for the same reason, $2(m'^2+p'^2)$, must express the perimeter of the triangle A′C′B′, and $(m'n' \div p'^2)(p'^2-n'^2)$ the radius of its inscribed circle. Now the perimeters of the two triangles are to be equal to each other, and also the radii of their inscribed circles, that is,

$p^2+m^2 = p'^2+m'^2$, and $(mn \div p^2)(p^2-n^2) = (m'n' \div p'^2)(p'^2-n'^2)$, Take $m' = m$, then $p' = p$, and the last of these equations, after being properly ordered, becomes $n(p^2 - n^2) = n'(p'^2 - n'^2)$: whence $p^2 = (n^3 - n'^3) \div (n - n') = n^2 + nn' + n'^2$. Put $p = an \div b - n'$; then $p^2 = n^2 + nn' + n'^2 = (an \div b - n')^2 = a^2n^2 \div b^2 - 2ann' \div b + n'^2$, which gives $n' = n(a^2 - b^2) \div (2ab + b^2)$: take $n = 2ab + b^2$, then $n' = a^2 - b^2$, $p = p' = an \div b - n' = a^2 + ab + b^2$, $q = mn \div p = m(2ab + b^2) \div (a^2 + ab + b^2)$, $q' = m'n' \div p' = m(a^2 - b^2) \div (a^2 + ab + b^2)$:

and therefore $AC = m^2 + n^2 = m^2 + (2ab + b^2)^2$, $BC = p^2 + q^2 = (a^2 + ab + b^2)^2 + m^2(2ab + b^2)^2 \div (a^2 + ab + b^2)$, $AB = m^2 - n^2 + p^2 - q^2 = m^2 - (2ab + b^2) + (a^2 + ab + b^2)^2 - m^2(2ab + b^2)^2 \div (a^2 + ab + b^2)^2$, $CD = 2mn = 2m(2ab + b^2)$.

Also in the triangle $A'C'B'$, $A'C' = m'^2 + n'^2 = m^2 + (a^2 - b^2)^2$, $B'C' = p'^2 + q'^2 = (a^2 + ab + b^2)^2 + m^2(a^2 - b^2)^2 \div (a^2 + ab + b^2)^2$, $A'B' = m'^2 - n'^2 + p'^2 - q'^2 = m^2 - (a^2 - b^2)^2 + (a^2 + ab + b^2)^2 - m^2(a^2 - b^2)^2 \div (a^2 + ab + b^2)^2$; $C'D' = 2m'n' = 2m(a^2 - b^2)$, where a, b, and m may be taken at pleasure. But in order to have whole numbers for the sides of the triangle, it is evident that m must be some multiple of $a^2 + ab + b^2$.

Example. Take $a = 2$, and $b = 1$, then $n = 2ab + b^2 = 5$, $n' = a^2 - b^2 = 3$, $p = p' = a^2 + ab + b^2 = 7$, $q = mn \div p = 5m \div 7$, and $q' = m'n' \div p' = 3m \div 7$. In order to have whole numbers for the sides, in the present circumstance, it is necessary to take m some multiple of 7; take $m = 14$, then $q = 5m \div 7 = 10$, and $q' = 3m \div 7 = 6$; whence $AC = m^2 + n^2 = 196 + 25 = 221$, $BC = p^2 + q^2 = 49 + 100 = 149$, $AB = m^2 - n^2 + p^2 - q^2 = 120$, and $CD = 2mn = 140$. Also $A'C' = m'^2 + n'^2 = 196 + 9 = 205$, $B'C' = p'^2 + q'^2 = 49 + 36 = 85$, $A'B' = m'^2 - n'^2 + p'^2 - q'^2 = 200$, and $C'D' = 2m'n' = 84$.

PROBLEM II.

To determine a polygon of any assigned number of sides, that may be inscribed in a circle, and such, that the sides of all the triangles, into which the polygon can possibly be divided, may all of them be expressed by rational numbers; and also, that the areas of the said triangles may every one of them be expressed by rational numbers.

SOLUTION.

Let ABCDE, (fig. 23, pl. C,) be a figure of five sides that may be inscribed in a circle, and draw lines between every two angles thereof: Let BE, BD cut AC in the points M and N respectively, and draw BF perpendicular to AC.

Then by the well known property of the circle,
$\text{ME} = \text{AM} \times \text{MC} \div \text{BM}$ and $\text{ND} = \text{AN} \times \text{NC} \div \text{BN}$.
The triangles MCB, MEA, are similar, as also the triangles NBA, NCA,

therefore $\text{BM} : \text{AM} :: \text{BC} : \text{AE} = \text{AM} \times \text{BC} \div \text{BM}$
and $\text{BN} : \text{NC} :: \text{AB} : \text{CD} = \text{NC} \times \text{AB} \div \text{BN}$:

The triangles MAB, MEC; NCB, NDB are also similar,
therefore $\text{BM} : \text{MC} :: \text{AB} : \text{CE} = \text{MC} \times \text{AB} \div \text{BM}$,
and $\text{BN} : \text{AN} :: \text{BC} : \text{AD} = \text{AN} \times \text{BC} \div \text{BN}$.

Again, by a known property of a trapezium inscribed in a circle $\text{ED} = (\text{BE} \times \text{AD} - \text{BD} \times \text{AE}) \div \text{AB}$.

From whence it is evident enough, when duly considered, that if the lines BA, BM, BN, and BC, together with AF, MF, CF, and NF are denoted by rational numbers, then the sides of all the triangles into which the figure ABCDE can possibly be divided will be expressed by rational numbers. We shall now show that the areas of all the said triangles, will be expressed by rational numbers, when, besides the before mentioned conditions, the perpendicular BF is denoted by a rational number. The area of any triangle, inscribed in a circle, is equal to the continued product of the three sides thereof divided by twice the diameter of the circumscribing circle, as may be easily proved.

Now $\text{BA} \times \text{BC} \div \text{BF} =$ the diameter of the circle circumscribing the figure ABCDE, and this expression for the diameter will be rational when BA, BC, and BF are so; consequently the area of each of the triangles, into which the figure ABCDE can be divided, being the continued product of three rational numbers divided by another rational number must be rational. We shall now proceed to apply what has been deduced to the obtaining of numerical values for the sides and diameter of the circumscribing circle of a five-sided figure answering the conditions of the problem.

Put $\text{AB} = m^2 + n^2$, $\text{AF} = m^2 - n^2$; $\text{BM} = m'^2 + n'^2$, $\text{MF} = m'^2 - n'^2$; $\text{BN} = m''^2 + n''^2$, $\text{NF} = m''^2 - n''^2$; $\text{BC} = m'''^2 + n'''^2$, and $\text{CF} = m'''^2 - n'''^2$; then $\text{BF} = 2mn = 2m'n' = 2m''n'' = 2m'''n'''$, or $\frac{1}{2}\text{BF} = mn = m'n' = m''n'' = m'''n'''$, which indicates that the number which denotes $\frac{1}{2}\text{BF}$ is to be resolved into two factors four different ways. Admitting fractional factors, any number whatever may be resolved into two factors in an infinite variety of ways, from whence we may infer, that the question admits of an infinite variety of answers. Perhaps the following assumption will give the least whole numbers possible for the sides of the five sided figure, and the diameter of the circumscribing circle.

Assume $mn = 24 \times 1 = 8 \times 3 = 6 \times 4 = 12 \times 2$; then we may evidently take

$m = 24, n = 1$; $m' = 8, n' = 3$; $m'' = 6, n'' = 4$; $m''' = 12, n'' = 2$; whence $AB = m^2 + n^2 = 577$, $AF = m^2 - n^2 = 575$: $BM = m'^2 + n'^2 = 73$, $MF = m'^2 - n'^2 = 55$, $BN = m''^2 + n''^2 = 52$, $NF = m''^2 - n''^2 = 20$, $BC = m'''^2 + n'''^2 = 148$, $CF = m'''^2 - n'''^2 = 140$,

$$ME = \frac{AM \times MC}{BM} = \frac{520 \times 195}{73}, BE = BM + ME = \frac{106729}{73},$$

$$ND = \frac{AN \times NC}{BN} = \frac{595 \times 120}{52}, BD = BN + ND = \frac{74104}{52},$$

$$AE = \frac{AM \times BC}{BM} = \frac{76960}{73}; CD = \frac{NC \times AB}{BN} = \frac{69240}{52},$$

$$CE = \frac{MC \times AB}{BM} = \frac{112515}{73}, AD = \frac{AN \times BC}{BN} = \frac{88060}{52},$$

$$ED = \frac{BE \times AD - BD \times AE}{AB} = \frac{6404700}{73 \times 52},$$ and the diameter of the circumscribing circle $\frac{AB \times BC}{BF} = \frac{577 \times 148}{48}$ $= \frac{21349}{12}$.

Integer values of the various parts of the figure may be readily obtained from those already found, viz. by reducing the said parts to a common denominator, and rejecting it. If what has been done is properly understood, there can remain no other difficulty in extending the question to a figure of a greater number of sides, but the unavoidable trouble of the calculation.

ARTICLE XV.

Two letters from Mr. Thomas Knight, *of Papcastle, on the proportionality of the force to the velocity, and the composition of forces.*

LETTER I.

On the proportionality of the force to the velocity.

To the Editor of the Mathematical Repository.

SIR,

I wish you could give the following paper in your next number.

Papcastle, I am Sir, your most obedient Servant,
July 8th, 1808. THOS. KNIGHT.

Is the proportionality of the force to the velocity, a necessary or contingent truth?

" Si ce principe est de vérité nécessaire, nous avouerons que les preuves qu'on en a aportées jusqu'ici ne nous paroissent pas hors d'atteinte." D'Alembert.

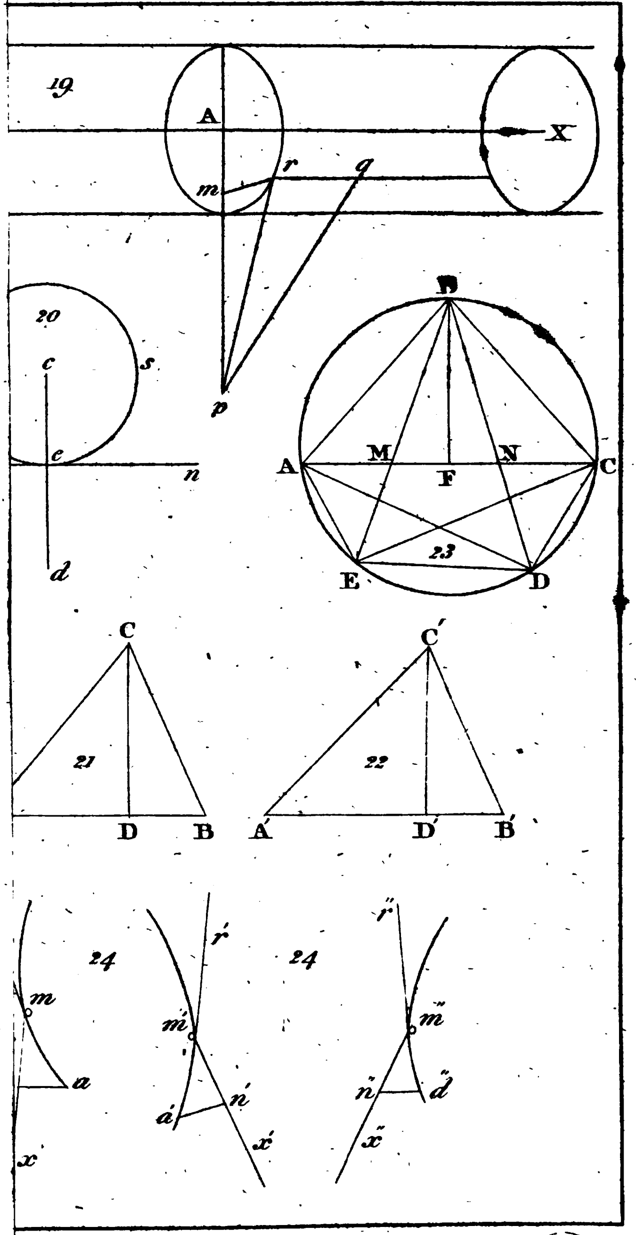
What is given for the composition of forces in most of our elementary books, is merely that of velocities, a matter which has no difficulty; but nothing can from thence be concluded respecting forces, till it be shewn, either by demonstration or experiment, what relation the force has to the velocity. There appears no reason why the one should be simply as the other. Might not a double force produce four times or eight times the motion? Indeed it implies (at first view) no contradiction to suppose the one any function whatever of the other. This being considered, many ingenious persons have investigated what shall be the force equivalent to two others, independently of any consideration of velocity. The two following propositions must be regarded as perfectly distinct, and have been so often proved, that I shall take them for granted.

1st. If there be impressed on a body a velocity AD (fig. 25, pl. D.) and another AB in a direction perpendicular to that of the former, the body will move in the diagonal AC, and with a velocity equal to that line.

2d. The effect of two forces AD, AB whose directions form

a right-

Vol II Plate C Fig 19 to 24



a right-angle, is equal to the effect of a force which acts in the direction of, and equals in quantity, the diagonal AC.*

I now come to the object of this paper; M. La Place affirms (see Méc. cél. Tome 1, page 15 to 18: also Syst. du Monde, p. 139, 140) that we cannot know the relation of the force to the velocity it generates, without having recourse to experience, and he shews by an ingenious analysis what may be the experiments proper to determine that relation: from them it appears that the force is as the velocity. This according to him is one of those facts—'*That Mecanics borrows of experience*;' and which reason alone could not have taught us;—'*considering our ignorance of the nature of moving force.*' I confess I am truly astonished that such should be his opinion; it seems to me on the contrary, that the proportionality of the force to the velocity is easily proved, that it is a *necessary truth*, and discovered without any aid from experience. For let Aβ, Aδ represent forces, *whose directions form a right angle*, giving separately the velocities AB, AD, or by the composition of *velocities*, giving the velocity AC in the direction AC. The resulting force (which by the composition of *forces* is the diagonal Aγ) is that which shall have the same effect as the composing ones: it must therefore give the velocity AC in the direction AC. Now if we suppose the force is not as the velocity, what is the consequence? That the ratio of AB to BC is different from that of Aβ to βγ, consequently that Aγ and AC differ in direction: whence this notable conclusion, that the direction of the motion is different from that in which the action of the force is exercised; this being absurd, the proposition I proposed to demonstrate is evident.

I cannot see any error in the above proof, yet I offer it with diffidence for this reason, that so plain a matter, if true, could hardly have escaped the sagacity of such men as D'Alembert and La Place.†

LETTER II.

On the composition of forces.

To the Editor of the Mathematical Repository.

SIR,

I endeavoured, in a former letter, to shew that there was only one relation, *mathematically possible*, between force and velocity. The composition of forces had been investigated by many eminent men, and I had at that time nothing to add on the subject: a demonstration, however, has since occured to me, which seems simpler than any hitherto given.

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* See Letter II.

† Note. I have taken nothing for granted in this paper. That the effect of a force in any direction will not be altered by another acting *at right angles* to that direction, I suppose no one will dispute. This is considered as an axiom by *Francœur*, whose work was corrected by *La Place*.

In shewing what is the *quantity* of the resulting force, I follow M. La Place, Méc. cél., but find the *direction*, which is the more difficult matter, by a much easier way, and without the assistance of the Integral Calculus.

Let x and y be two forces acting on a point, and forming a right-angle with each other, z the resulting force, and θ the angle formed by x and z. I shall suppose the first part demonstrated as by La Place; viz. that $x = z\varphi(\theta)$, $y = z\varphi(\frac{\pi}{2} - \theta)$, $x^2 + y^2 = z^2$ .... (α), φ being some unknown function. Because $\frac{x}{z}$ is a function of θ, it follows conversely that θ and consequently $\cos\theta$ are functions of $\frac{x}{z}$, or $\cos\theta = \psi(\frac{x}{z})$; in like manner $\cos(\frac{\pi}{2} - \theta) = \psi(\frac{y}{z})$. Let these functions be supposed expanded as follows

$$\left.\begin{aligned}\cos\theta &= {}^{\prime}A\frac{x}{z} + {}^{\prime\prime\prime}A\frac{x^3}{z^3} + {}^{\prime\prime\prime\ldots5}A\frac{x^5}{z^5} + \&c.\\ \cos(\frac{\pi}{2} - \theta) &= {}^{\prime}A\frac{y}{z} + {}^{\prime\prime\prime}A\frac{y^3}{z^3} + {}^{\prime\prime\prime\ldots5}A\frac{y^5}{z^5} + \&c.\end{aligned}\right\}$$

It is plain there can be no even powers.

By squaring each of these equations and adding them together, there arises $\cos^2\theta + \cos^2(\frac{\pi}{2} - \theta) =$

$$1 = {}^{\prime}A\frac{x^2 + y^2}{z^2} + 2{}^{\prime}A\,{}^{\prime\prime\prime}A\frac{x^4 + y^4}{z^4} + \left.\begin{aligned}&{}^{\prime\prime\prime}A^2\\ &+ 2{}^{\prime}A\,{}^{\prime\prime\prime\ldots5}A\end{aligned}\right\}\frac{x^6 + y^6}{z^6}$$

$$\left.\begin{aligned}&+ 2{}^{\prime\prime\prime}A\,{}^{\prime\prime\prime\ldots5}A\\ &+ 2{}^{\prime}A\,{}^{\prime\prime\prime\ldots7}A\end{aligned}\right\}\frac{x^8 + y^8}{z^8}\left.\begin{aligned}&+ {}^{\prime\prime\prime\ldots5}A^2\\ &+ 2{}^{\prime\prime\prime}A\,{}^{\prime\prime\prime\ldots7}A\\ &+ 2{}^{\prime}A\,{}^{\prime\prime\prime\ldots9}A\end{aligned}\right\}\frac{x^{10} + y^{10}}{z^{10}} + \&c. \ldots(\beta).$$

Now (α) gives $1 = \frac{x^2 + y^2}{z^2}$, and this equation can be no other than (β). For if it were possible that there could be two different relations between x, y and z, we might, by elimination, get an equation between z and x, that is, z would be a function of x *alone*, which is absurd. By comparing then these two equations, we find

$${}^{\prime}A = 1, 2{}^{\prime}A\,{}^{\prime\prime\prime}A = 0, {}^{\prime\prime\prime}A^2 + 2{}^{\prime}A\,{}^{\prime\prime\prime\ldots5}A = 0, 2{}^{\prime\prime\prime}A\,{}^{\prime\prime\prime\ldots5}A + 2{}^{\prime}A\,{}^{\prime\prime\prime\ldots7}A = 0, {}^{\prime\prime\prime\ldots5}A + 2{}^{\prime\prime\prime}A\,{}^{\prime\prime\prime\ldots7}A + 2{}^{\prime}A\,{}^{\prime\prime\prime\ldots9}A = 0, \&c.$$

and from these we get ${}^{\prime\prime}A = 0$, $\overset{\prime\prime\ldots5}{A} = 0$, $\overset{\prime\prime\prime\ldots7}{A} = 0$, &c.
therefore $\cos\theta = \frac{x}{z}$, which was the thing to be found.

Scholium. As Mr. La Place's works may not be in the hands of all your readers, I shall give his demonstration, of the first part of the composition of forces, alluded to in the above paper.

Let us suppose at first the forces x and y infinitely small and equal to dx and dy; let us suppose next, that x becoming successively dx, $2dx$, $3dx$, &c. y becomes dy, $2dy$, $3dy$, &c. it is plain that the angle θ will always be the same, and that the resulting force z will become successively dz, $2dz$, $3dz$, &c. thus in the successive increments of x, y and z, the ratio of x to z will be constant, and may be expressed by a function of θ, which we will denote by $\varphi(\theta)$; we shall have then $x = z.\varphi(\theta)$, in which equation we may change x into y, provided that we also change θ into $\frac{\pi}{2} - \theta$, π being the semi-circumference whose radius is unity. Now we may consider the force x as resulting from two forces x' and x'', the first acting in the direction of z, and the second perpendicular to that direction. The force x which results from these two new forces, making the angle θ with the force x' and the angle $\frac{\pi}{2} - \theta$ with the force x'', we shall have

$x' = x\varphi(\theta) = \frac{x^2}{z}$; $x'' = x\varphi\left(\frac{\pi}{2} - \theta\right) = \frac{xy}{z}$; we may therefore substitute these two forces in the place of x. We may in like manner put instead of y, two new forces y' and y'', the first of which is equal to $\frac{y^2}{z}$ and in the direction of z, and the other is $\frac{xy}{z}$ and perpendicular to z; we have thus in place of the two forces x and y, the four following

$$\frac{x^2}{z},\ \frac{y^2}{z},\ \frac{xy}{z},\ \frac{xy}{z}.$$

The two last acting in opposite directions destroy one another; the others, acting in the same direction, added together will form the resulting force z; we shall have then $x^2 + y^2 = z^2$.

Papcastle,
October 23, 1808.

I am Sir, Your Obedient Servant,
Thos. Knight.

ARTICLE

ARTICLE XVI.

On the Composition of rotatory Motions, By *Mr.* KNIGHT.

To the Editor of the Mathematical Repository.

SIR,

In this paper I have demonstrated in an easy manner the propositions that have been given by *Frisi* and *La Grange*, concerning the composition of rotatory motions; to which I have added a still more general theorem.

Let a body have forces impressed on it, that would separately give it a movement of rotation, about the axis Cx, (fig. 28, pl. D.) with the angular velocity v; and about Cy with the angular velocity v'. It is plain that a particle in the line Cy will not be affected in its motion about Cx, by the other rotation about Cy. The same observation (mutatis mutandis) may be applied to a particle in the line Cx: but in any intermediate line in the plane yCx, as Cm, the motion round the one axis will impede that round the other; and if we can find a line in which all the particles are at rest, *that* will be the momentary axis of rotation composed of and equivalent to the two others.

Put Cm = r, angle xCm = θ, angle yCm = θ',
A perpendicular from m on Cx will be $r \sin\theta$,
A perpendicular from m on Cy will be $r \sin\theta'$, and it is plain that m will be at rest if

$$v' : v = r\sin\theta : r\sin\theta' = \sin\theta : \sin\theta' \quad \ldots\ldots\ldots\ldots (\alpha),$$

which determines the situation of the momentary axis.

To find the velocity round this, let us suppose ymx to be a circle drawn through m, the centre being C. Let ay be the velocity of the point y; we have $ay = vr \sin x\mathrm{C}y$. Now the velocity about Cm at the distance r is plainly $ay \times \frac{r}{r\sin\theta'} = \frac{ay}{\sin\theta'}$, consequently the velocity at distance unity, or the

$$\text{angular vel.} = \frac{ay}{r\sin\theta'} = v \times \frac{\sin x\mathrm{C}y}{\sin\theta'} \quad \ldots\ldots\ldots\ldots\ldots (\beta).$$

The theorems in equations (α) and (β) were first given by *Frisi*. If, besides the two rotations already mentioned, there had been given, at the same time, another round a third axis Cz, with an angular velocity $= v''$; it is plain that the axis of rotation, arising from these three, would be in the plane zCm, as at Cm', suppose it to decline from Cm by the angle θ'', and from Cz by θ'''; we find, as we did (α) and (β),

$$v'' : v \times \frac{\sin xCy}{\sin \theta'} = \sin \theta'' : \sin \theta''' \quad \ldots\ldots\ldots\ldots (\gamma)$$

and angular velocity $= v \times \dfrac{\sin xCy}{\sin \theta'} \times \dfrac{\sin zCm}{\sin \theta'''} \ldots\ldots\ldots (\delta)$.

If, moreover, a rotation had been given round a fourth axis Cu, with an angular velocity $= v'''$, we should have

$$v''' : v \times \frac{\sin xCy \,.\, \sin zCm}{\sin \theta' \,.\, \sin \theta'''} = \sin \theta'''' : \sin \theta''''' \quad \ldots\ldots (\varepsilon),$$

θ'''' and θ''''' being angles that the momentary axis declines from Cm' and Cu; and the angular velocity $=$

$$v \times \frac{\sin xCy \,.\, \sin zCm \,.\, \sin uCm'}{\sin \theta' \,.\, \sin \theta''' \,.\, \sin \theta''''},$$

and thus we may proceed to any number of axes.

We will now consider the case in which the angles xCy, zCm, uCm' &c. are right, and their sines unity. We have $\sin \theta = \cos \theta'$, $\sin \theta'' = \cos \theta'''$ &c. here (α) becomes $v' : v = \cos \theta' :$ $\sin \theta' = \surd(1 - \sin^2 \theta') : \sin \theta'$; whence $\sin \theta' = \dfrac{v}{\surd(v^2 + v'^2)}$.

We get in like manner from (γ), $\sin \theta''' = \dfrac{\surd(v^2 + v'^2)}{\surd(v^2 + v'^2 + v''^2)}$, and from ($\varepsilon$) $\sin \theta''''' = \dfrac{\surd(v^2 + v'^2 + v''^2)}{\surd(v^2 + v'^2 + v''^2 + v'''^2)}$, and so on: and if the number of axes be n, and the velocities round those axes v, v', v'', ... $v^{'''\ldots n}$, the angular velocity round the axis arising from the composition of them all will be $\surd\{v^2 + v'^2 + v''^2 + \ldots\ldots + (v^{'''\ldots n})^2\}$. Of this very general theorem the case considered by La Grange of three axes *perpendicular to each other*, is a particular case, in which ang vel. $= \surd(v^2 + v'^2 + v''^2)$ and $\sin \theta' = \dfrac{v}{\surd(v^2 + v'^2)}$, $\sin \theta''' =$ $\dfrac{\surd(v^2 + v'^2)}{\surd(v^2 + v'^2 + v''^2)}$, or, the angle zmy being a right one, we easily find that Cm' makes angles with Cx, Cy, Cz whose cosines are

$$\frac{v}{\surd(v^2 + v'^2 + v''^2)}, \quad \frac{v'}{\surd(v^2 + v'^2 + v''^2)}, \quad \frac{v''}{\surd(v^2 + v'^2 + v''^2)}).$$

In the general case I treated of above, the angles amy, $um'm$, &c. are any whatsoever, and if their magnitude be given, the situation of the composed axis is also given, by means of the values of $\sin\theta'$, $\sin\theta'''$, $\sin\theta'''''$, &c.

I am Sir,

Papcastle,
Dec. 20, 1808.

Your obedient Servant,
THOS. KNIGHT.

ARTICLE XVII.

On the Expansion of certain Functions. *By Mr.* KNIGHT.

To the Editor of the Mathematical Repository.

SIR,

It seems not a little extraordinary that the following demonstrations should have escaped the notice of mathematicians, at least as far as I know.

To expand the exponential a^x.

Let $a^x = 1 + {'A}x + {''A}x^2 + {'''A}x^3 +$ &c.

Its fluxion gives

$$a^x = \frac{'A}{L.a} + \frac{2\,''Ax}{L.a} + \frac{3\,'''Ax^2}{L.a} + \frac{4\,''''Ax^3}{L.a} + \&c.$$

Whence ${'A} = L.a$, ${''A} = \frac{(L.a)^2}{1.2}$, ${'''A} = \frac{(L.a)^3}{1.2.3}$, &c.

Let $a^{c+c'x+c''x^2} = A + {'A}x + {''A}x^2 + {'''A}x^3 +$ &c.

Then $a^{c+c'x+c''x^2}(c'+2c''x) = \frac{'A}{L.a} + \frac{2\,''Ax}{L.a} + \frac{3\,'''Ax^2}{L.a} +$ &c.

Multiply the first by $c'+2c''x$, and there arises

$$a^{c+c'x+c''x^2}.(c'+2c''x) = c'A + c'\,{'A} \times x + c'\,{''A} \times x^2 + \&c.$$
$$\qquad\qquad + 2c''A \qquad + 2c''\,{'A}$$

Whence by comparing the two last

$\frac{'A}{L.a} = c'A$, $\frac{2\,''A}{L.a} = c'\,{'A} + 2c''A$, $\frac{3\,'''A}{L.a} = c'\,{''A} + 2c''\,{'A}$, &c.

But $A = a^c$, consequently all the other coefficients are known.

Let $\sin x = x + {}'''Ax^3 + {}'''^{\cdots 5}Ax^5 +$ &c. the fluxional coefficient of the second order gives

$$\sin x = -2.3\,{}'''Ax - 4.5\,{}'''^{\cdots 5}Ax^3 + \&c.$$

Whence $${}'''A = -\frac{1}{2.3},\ {}'''^{\cdots 5}A = \frac{1}{2.3.4.5}, \&c.$$

In like manner is cos x found from its second fluxion; but both might have been found at once by means of their *first fluxions only;* in the manner following:

Let $\sin(w+x) = \sin w + {}'Ax + {}''Ax^2 + {}'''Ax^3 + \&c.$
and $\cos(w+x) = \cos w + {}'Bx + {}''Bx^2 + {}'''Bx^3 + \&c.$
Then $\cos(w+x) = {}'A + 2\,{}''Ax + 3\,{}'''Ax^2 + 4\,{}''''Ax^3 + \&c.$
and $\sin(w+x) = -{}'B - 2\,{}''Bx - 3\,{}'''Bx^2 - 4\,{}''''Bx^3 + \&c.$
From the two middle equations we find
${}'A = \cos w,\ {}'B = 2\,{}''A,\ {}''B = 3\,{}'''A,\ {}'''B = 4\,{}''''A,$ &c.
from the extreme ones

$${}'B = -\sin w,\ {}''B = -\frac{{}'A}{2},\ {}'''B = -\frac{{}''A}{3}\ \&c.$$

Whence all the coefficients are known.

With equal ease are found the sine and cosine of a multinomial; I shall here confine myself to $a + a'x + a''x$. Let then
(α) $\sin(a+a'x+a''x^2) = A + {}'Ax + {}''Ax^2 + {}'''Ax^3 + \&c.$
(β) $\cos(a+a'x+a''x^2) = B + {}'Bx + {}''Bx^2 + {}'''Bx^3 + \&c.$
The fluxional coefficients of the first order are
(γ) $\cos(a+a'x+a''x^2)(a'+2a''x) = {}'A + 2\,{}''Ax + 3\,{}'''Ax^2 + \&c.$
(δ) $\sin(a+a'x+a''x^2)(a'+2a''x) = -{}'B - 2\,{}''Bx - 3\,{}'''Bx^2 - \&c.$

(α) and (β) multiplied by $a' + 2a''x$ produce

$$(\varepsilon)\ \sin(a+a'x+a''x^2)(a'+2a''x) = a'A + \left.\begin{matrix}a'\,{}'A\\ +2a'\,{}'A\end{matrix}\right\} x + \left.\begin{matrix}a'\,{}''A\\ +2a''\,{}'A\end{matrix}\right\} x^2 + \&c.$$

$$(\zeta)\ \cos(a+a'x+a''x^2)(a'+2a''x) = a'B + \left.\begin{matrix}a'\,{}'B\\ +2a'\,{}'B\end{matrix}\right\} x + \left.\begin{matrix}a'\,{}''B\\ +2a''\,{}'B\end{matrix}\right\} x^2 + \&c.$$

Compare (δ) and (ε) and there arises

$$a'A = -{}'B,\ \frac{a'\,{}'A}{2} + a''A = -{}''B,\ \frac{a'\,{}''A}{3} + \frac{2a''\,{}'A}{3} = -{}'''B\ \&c.$$

From (γ) and (ζ) we find

$$a'B = {}'A,\ \frac{a'\,{}'B}{2} + a''B = {}''A,\ \frac{a'\,{}''B}{3} + \frac{2a''\,{}'B}{3} = {}'''A, \&c.$$

But $A = \sin a$, $B = \cos a$, whence all the coefficients are known. Thus the sine and cosine of a multinomial are found at the same time, and by a simpler method than any I have seen.

In a manner not very different, may the elliptick sine and cosine be found at *one operation*, in terms of the elliptick arc; from whence the sine and cosine of the circle are deduced as a particular case, which, though of no great use; I shall add here on account of its novelty.

AM′ (fig. 29, pl. D.) is an ellipsis, AM a circle whose centre is that of the ellipsis, and its radius AC half the transverse diameter: let this $= 1$, and the semi-conjugate $= m$. PM is perpendicular to AC. Put $AM = x$, $A'M = x'$:
$P'M$ is the elliptick sine of $x' = m$ sine of x, by conics;
CP is the elliptick cosine of $x' =$ cos x. Let

$$\text{elliptick} \sin x' = m \sin x = x' + {}^{\prime\prime\prime}Ax'^3 + {}^{\prime\prime\prime\ldots5}Ax'^5 + {}^{\prime\prime\prime\ldots7}Ax'^7 + \&c$$

$$\text{elliptick} \cos x' = \cos x = 1 + {}^{\prime\prime}Bx'^2 + {}^{\prime\prime\prime\prime}Bx'^4 + {}^{\prime\prime\prime\ldots6}Bx'^6 + \&c.$$

whose fluxions are

$$m \cos x\dot{x} = \dot{x}'(1 + 3\,{}^{\prime\prime\prime}Ax'^2 + 5\,{}^{\prime\prime\ldots5}Ax'^4 + 7\,{}^{\prime\prime\prime\ldots7}Ax'^6 + \&c.$$

$$\sin x\dot{x} = -\dot{x}'(2\,{}^{\prime\prime}Bx' + 4\,{}^{\prime\prime\prime\prime}Bx'^3 + 6\,{}^{\prime\prime\ldots6}Bx'^5 + \&c.$$

By the nature of the circle $\sin^2 x + \cos^2 x = 1$; and in all curves $\overline{\dot{AM}}^2 = \overline{\dot{AP}}^2 + \overline{\dot{PM}}^2$ which is in the present case $(\dot{x}')^2 = \sin^2 x\dot{x}^2 + m \cos^2 x\dot{x}^2$.

If we substitute in these two equations the above expansions, there arises

$$1 = 1 + \left.\begin{matrix} \frac{1}{m^2} \\ +2B'' \end{matrix}\right\} x'^2 + \left.\begin{matrix} \frac{2\,{}^{\prime\prime\prime}A}{m^2} \\ +\ B'' \\ +2B'''' \end{matrix}\right\} x'^4 + \left.\begin{matrix} \frac{{}^{\prime\prime\prime}A^2}{m^2} \\ +\frac{2\,{}^{\prime\prime\ldots5}A}{m^2} \\ +2B''B'''' \\ +2B''^{\ldots6} \end{matrix}\right\} x^6 + \&c. \quad \text{And}$$

$$1 = 1 + \left.\begin{matrix} 2.3\ {}^{\prime\prime\prime}A \\ +2.2\,B'^2 \end{matrix}\right\} x'^2 + \left.\begin{matrix} 3.3\ {}^{\prime\prime\prime}A^2 \\ +2.5\ {}^{\prime\prime\ldots5}A \\ +2.2.4B''B'''' \end{matrix}\right\} x'^4 + \left.\begin{matrix} 2.3.5\ {}^{\prime\prime\prime}A\ {}^{\prime\prime\ldots5}A \\ +2.7\ {}^{\prime\prime\ldots7}A \\ +4.4\ B''^2 \\ +2.2.6\,B''\ B''^{\ldots6} \end{matrix}\right\} x^6 + \&c.$$

from which, by making separately equal to nothing the coefficients of the powers of x in each, we get two sets of equations, whence are derived the coefficients ${}^{\prime\prime\prime}A$, ${}^{\prime\prime\ldots5}A$, &c. B', B''', &c. and

$$\text{elliptick sin of } x' = x' - \frac{1}{2.3.m^4}x'^3 + \frac{13 - 12m^2}{2.3.4.5.m^8}x'^5 - \&c.$$

$$\text{elliptick cos of } x' = 1 - \frac{1}{2m^2}x^2 + \frac{4 - 3m^2}{2.3.4m^6}x^4 - \&c.$$

By making $m = 1$, we get the common series for the sine and cosine of a circular arc: And, by reverting the series, the rectification of an elliptick arc is accomplished, which though it may not come out in a very useful form, is so far remarkable, that it has been (for the first time I believe) found without help from the integral calculus.

ARTICLE XVIII.

Two letters from Mr. KNIGHT, *on the expansion of any Function of a Multinomial.*

LETTER I.

To the Editor of the Mathematical Repository,

Sir,

The following paper contains an investigation of the expansion of any function of a multinomial, which has hitherto received no demonstration but that given by *M. Arbogast*, in his very learned work, '*Du calcul des dérivations*'. As the method I use is quite different from his, I think it may not be uninteresting to the learned part of your readers.

Papcastle, I am Sir, your's, &c.
Nov. 5, 1808. THOS. KNIGHT.

It is well known to mathematicians that, if u be a function of a', a'', a''', &c. and expanded according to the powers and products of those quantities, the coefficient of $a'^{m'}a''^{m''}a'''^{m'''}$&c. will be

$$\frac{\left(\dfrac{\overset{m'+m''+m'''+}{\overline{\dot{u}}}}{\dot{a}'^{m'}\,\dot{a}''^{m''}\,\dot{a}'''^{m'''}\ \text{\&c.}}\right)}{1.2.3\ldots m'\times 1.2.3\ldots m''\times 1.2.3\ldots m'''\times\text{\&c.}}\ \ldots\ldots\ldots\ (\alpha)$$

provided that in this expression we make a', a'', a''', &c. vanish after the indicated fluxional operation. I shall apply this to the following form, $f(A + c'a' + c''a'' + c'''a''' +)$; where a', a'', a''', &c. represent x, x^2, x^3, &c. being more manageable for the present. I shall denote by $'f$, $''f$, $'''f$, &c. the successive fluxional coefficients of the function f.

The first thing to be observed is that the expression (α), when the quantities a', a'', a''', &c. have been made to vanish, will become

$$
\frac{{}^{m}\ldots m' + m'' + m''' + \atop f(A)\, c'^{m'}\, c''^{m''}\, c'''^{m'''}\ \&c.}{1.2.3\ldots m' \times 1.2.3\ldots m'' \times 1.2.3\ldots m''' \times \&c.} \ldots\ldots\ldots (\beta).
$$

Now let it be required to find the coefficient of any power (*n*) of x in the expansion of the above function.

Place in a horizontal line the successive powers of a' multiplied by one of the other quantities, so that the sum of the *strokes* shall be *n*. In a second line the powers of a' combined with two other quantities: and so on. Then proceed in the same manner with a'' &c. Observe to let those quantities stand in the same perpendicular line, the sum of whose *exponents* forms the same number.

To illustrate this, let *n* be 6; by proceeding as above directed we form the following scheme;

a''''''	$a'a'''''$	a'^2a''''	a'^3a'''	a'^4a''	a'^6
		$a'a''a'''$	$a'^2a''^2$		
	$a''a''''$	a''^3			
	a'''^2				
$'f(A)$	$''f(A)$	$'''f(A)$	$''''f(A)$	$'''''f(A)$	$''''''f(A)$

It is plain that the sum of the coefficients of the quantities thus formed will be the coefficient of x^6. It appears immediately from formula (β), that each perpendicular column has all its coefficients multiplied by the fluxional coefficient placed underneath it; and by the application of the same formula to the above table, we find with the greatest facility that the coefficient of x^6 in the expansion of $f(A + c'x + c''x^2 + c'''x^3 +)$ is

$$
{}'f(A)c'''''' + {}''f(A)\left|\begin{array}{l} c'c''''' \\ +c''c'''' \\ +\dfrac{c'''^2}{1.2} \end{array}\right. + {}'''f(A)\left|\begin{array}{l} \dfrac{c'^2c''''}{1.2} \\ +c'c''c''' \\ +\dfrac{c''^3}{1.2.3} \end{array}\right. + {}''''f(A)\left|\begin{array}{l} \dfrac{c'^3c'''}{1.2.3} \\ +\dfrac{c'^2c''^2}{4} \end{array}\right. + {}'''''f(A)\frac{c'^4c''}{1.2.3.4} + {}''''''f(A)\frac{c'^6}{1.2.3.4.5}
$$

And in the same manner may the coefficient of any other power of x be found without any knowledge of those that go before it.

LETTER II.

To the Editor of the Mathematical Repository.

SIR,

I sent you lately a method of expanding any function of a multinomial, which however had the disadvantage of requiring the previous knowledge of another expansion. In *M. Arbogast's* method it was deduced from *Taylor's Theorem*, by a rather long process. I have been since a good deal surprised at finding that, notwithstanding the discovery was reserved for a mathematician of our times, it may be accomplished with *as much ease as the common fluxional demonstration of the binomial theorem.*

Let $f(a+x)$ represent any function of $a+x$. If the fluxion be taken with respect to a and x, the fluxional coefficient is plainly the same in both cases; because that coefficient is independent of the *magnitude* of the fluxion of $a+x$, and remains the same whether this latter be $\dot{a}+\dot{x}$, or $\dot{a}$ or $\dot{x}$. This is analytically expressed thus,

$$\left(\frac{\dot{\overline{f(a+x)}}}{\dot{a}}\right)=\left(\frac{\dot{\overline{f(a+x)}}}{\dot{x}}\right);$$

Hence it is plain, that

$$\int\left(\frac{\dot{\overline{f(a+x)}}}{\dot{a}}\right)\dot{x}=f(a+x).$$

This being premised, let

$$f(A+c'x+c''x^2+c'''x^3+)=B+B_1x+B_2x^2+\ldots+B_nx^n+\ldots(\alpha).$$

The first member may be considered as a function of a binomial, the first term being A, and second $c'x+c''x^2+$. Let B', B'_1, B'_2, &c. represent the fluxional coefficients of B, B_1, B_2, &c. with respect to A supposed variable, and we shall have

$$\frac{\dot{\overline{f(A+c'x+c''x^2+c'''x^3+)}}}{\dot{A}}=B'+B'_1x+B'_2x^2+\ldots B'_nx^n+$$

which multiplied by $c'\dot{x}+2c''x\dot{x}+3c'''x^2\dot{x}+\ldots nc'''^{\cdots n}x^{n-1}\dot{x}+$ will, after the fluent is taken, give $f(A+c'x+c''x^2+)$; as appears from what was above premised.

We shall have then, by comparison with (α),

$$\int c'B'\dot{x}=B_1x,\quad \int(2c''B'+c'B'_1)\,x\dot{x}=B_2x^2,$$

$$\int(3c'''B'+2c''B'_1+c'B'_2)\,x^2\dot{x}=B_3x^3,$$ &c. and by actually taking the fluents, the coefficients are expressed by the following recurring terms;

$$B_1=c'B'$$

$$B_2=\frac{2c''B'+c'B'_1}{2}$$

$$B_3=\frac{3c'''B'+2c''B'_1+c'B'_2}{3}$$

$$B_4 = \frac{4c''''B' + 3c'''B'_1 + 2c''B'_2 + c'B'_3}{4}$$

……………………………………

$$B_n = \frac{nc''\cdots^{n}B' + (n-1)\,c'''\cdots^{(n-1)}B'_1 + \ldots\ldots 2c''B'_{n-2} + c'B'_{n-1}}{n}$$

But $B = f(A)$ and $B' = \frac{\dot{\overline{f(A)}}}{\dot{A}}$, whence all the rest are known. It is plain that the use of fluents might have been avoided by taking the fluxion of equation (z).

Papcastle, I am Sir, Your Obedient Servant,
Decemb. 7, 1808. THOS. KNIGHT

ARTICLE XIX.

Demonstration of a Theorem in the Diophantine Analysis. By Mr. P. BARLOW, *of the Royal Military Academy, Woolwich.*

To the Editor of the Mathematical Repository.

SIR,

Should the enclosed paper appear to you to merit a place in your next number, its insertion will much oblige

Your's, &c.

August 1, 1808. P. BARLOW.

THEOREM.

Every integral number whatever, is either a square, or the sum of *two, three,* or *four* squares.

The above curious property of numbers forms a part of the celebrated theorem of Fermat, viz. "Every number is a triangular number, or the sum of *two,* or *three* triangular numbers; a square, or the sum of *two, three,* or *four* squares; a pentagonal, or the sum of *two, three, four,* or *five* pentagonal numbers; and so on."[†]

This theorem was left by Fermat without a demonstration, nor has it ever been generally demonstrated; although an attempt of that kind has lately been made in one of our monthly publications; the following demonstration, therefore, for the case of squares, may not be unacceptable to some of your English readers. It is not given as original, the only merit that can be claimed is, that of having simplified and extended the demonstration of Le Gendre, by the introduction of the two following propositions.

PROP. I.

If A be any prime number, and all the consecutive squares $1^2, 2^2, 3^2, 4^2$, &c. ...... $\left(\frac{A-1}{2}\right)^2$, be divided by A, they will each leave a different positive remainder.

For let r^2 and s^2 be any two of those squares, (each of which must necessarily be less than ½A, by the prop,) then if it be possible for them to leave the same remainder after being divided by A, it is evident that A will be a divsior of the difference of those squares, and we shall have $\frac{r^2 - s^2}{A} = \frac{(r+s)(r-s)}{A} = p$ an integral number; but this is impossible, for since $r \angle \frac{1}{2}A$, and $s \angle \frac{1}{2}A$, the factors $r + s$, and $r - s$ are each less than A, therefore as A is a prime number, and divides neither of the factors, it cannot divide their product; and consequently, no two of the positive remainders can be equal to each other.

Q. E. D.

Cor. 1. In the same manner it is proved, that the negative remainders are all different from each other.

Cor. 2. Hence also we may see in what cases the positive and negative remainders are equal to each other, for then, it is evident that A will be a divisor of the sum of those squares, and we shall have $\frac{r^2 + s^2}{A} = p$ an integral number. Therefore when A is not a divisor of the sum of two squares, the positive and negative remainders are all different from each other, and include all numbers from 1 to A — 1.

Cor. 3. When A is not a divisor of the sum of two squares, that is when all the positive and negative remainders are different from each other, then some of each of those remainders, will be *greater*, and some *less* than ½A. For all the consecutive squares under A will be found amongst the positive remainders, some of which must necessarily be *greater*, and some *less* than ½A, and since the positive and negative remainders, include all numbers from 1 to A — 1, the same, therefore, must evidently be true for the negative remainders.

PROP. II.

If A be any prime number, it is always possible to find four squares w^2, x^2, y^2, z^2; such that their sum is divisible by A; that is the equation $w^2 + x^2 + y^2 + z^2 = {}'AA$ is always possible.

When the given prime number A is a divisor of the sum of two squares, the proposition is evident; it will therefore only be necessary to consider the case in which A is not a divisor of the sum of two squares, and, consequently, when all the remainders of the consecutive squares are different from each other. Cor. 2. Prop. I. Now in this case, we shall find some of the positive remainders *greater*, and some *less* than $\frac{1}{2}$A; and the same of the negative remainders. Cor. 3. Prop. 1. It is, therefore, always possible to find two squares, such that each being divided by A, the positive remainder of one, shall exceed the negative remainder of the other, by unity: and also two others, such that each being divided as before, the negative remainder of one, shall exceed the positive remainder of the other, by unity; that is the equations $w^2 + x^2 - 1 = mA$, and $y^2 + z^2 + 1 = nA$ are always possible.

If this be not evident at first sight, it may be demonstrated thus; let p be the least negative remainder, then $p + 1$ must be found either amongst the positive or negative remainders; now if it be found amongst the positive, we have at once a positive remainder, which exceeds a negative one by unity; and if it be not found amongst the positive, then $p + 1$ is still a negative remainder, and $p + 2$ must be found either amongst the positive or negative remainders, if it be amongst the positive, we shall have a positive remainder, which exceeds a negative one by unity; but if not, then $p + 2$ is still negative and proceeding in the same manner with this last, we must necessarily, (as some of each of the remainders are *greater*, and some *less* than $\frac{1}{2}$A,) arrive at that negative remainder $p + q$, such that $p + q + 1$ shall be a positive remainder; and consequently the equation $w^2 + x^2 - 1 = mA$ is always possible; in the same manner we may demonstrate the possibility of the equation $y^2 + z^2 + 1 = nA$.

Having thus proved the truth of those equations, we have by addition and division $\frac{w^2 + x^2 + y^2 + z^2}{A} = m + n$, or $w^2 + x^2 + y^2 + z^2 = \text{'AA}$ .... *Q. E. D.*

PROP. III.

The product of a sum of four squares, by a sum of four squares, is likewise the sum of four squares.

This will be made evident by developing the following formulæ, which will be found identical; viz.

$$(w^2 + x^2 + y^2 + z^2) \times (w'^2 + x'^2 + y'^2 + z'^2)$$
$$= (ww' + xx' + yy' + zz')^2 + (wx' - xw' + yz' - zy')^2$$
$$+ (wy' - xz' - yw' + zx')^2 + (wz' + xy' - yx' - zw')^2$$

Q. E. D.

PROP.

PROP. IV.

Every prime number is the sum of *two*, *three*, or *four* squares.

It has been proved in Prop. 2, that the equation $w^2+x^2+y^2+z^2 = {}'AA$ is always possible, where it also appears, that the roots of each of those squares may be less than $\frac{1}{2}A$; the same may be otherwise shewn thus: Since A is a divisor of the formula $(w^2 + x^2 + y^2 + z^2)$, it is also a divisor of the formula $(w - \alpha A)^2 + (x - \beta A)^2 + (y - \gamma A)^2 + (z - \delta A)^2$, where the indeterminates α, β, γ, and δ may always be so assumed, that $(w - \alpha A)^2$, $(x - \beta A)^2$, &c. shall be less than $(\frac{1}{2}A)^2$; therefore in the following demonstration, whenever it is proved that any number A is a divisor of the sum of four squares, it is taken for granted, that it is also a divisor of four squares, each of which has its root less than $(\frac{1}{2}A)$. This being premised, the demonstration will be as follows:

Since $w^2 + x^2 + y^2 + z^2 = {}'AA$, and each of those roots are less than $\frac{1}{2}A$, we have ${}'AA \angle \frac{1}{4}A^2$, or ${}'A \angle A$. Now if ${}'A = 1$, it is evident that A will be equal to the sum of the four squares $w^2 + x^2 + y^2 + z^2$, and the proposition is demonstrated.

Let then ${}'A > 1$, and because ${}'A$ is a divisor of the formula $(w^2 + x^2 + y^2 + z^2)$, it is also a divisor of $(w - \alpha\,{}'A)^2 + (x - \beta\,{}'A)^2 + (y - \gamma\,{}'A)^2 + (z - \delta\,{}'A)^2$, where each of these roots are less than $\frac{1}{2}\,{}'A$.

If now we make $(w - \alpha\,{}'A)^2 + (x - \beta\,{}'A)^2 + (y - \gamma\,{}'A)^2 + (z - \delta\,{}'A)^2 = {}''A\,{}'A$, we shall have ${}''A\,{}'A \angle \frac{1}{4}\,{}'A^2$, or ${}''A \angle {}'A$.

Now if by means of the formula at Prop. 3, we multiply the value of ${}'AA$ by that of ${}''A\,{}'A$, we shall find for a product a sum of four squares, of which each is divisible by ${}'A^2$, and having performed this division, we shall have ${}''AA = (A - \alpha w - \beta x - \gamma y - \delta z)^2 + (\alpha x - \beta w + \gamma z - \delta y)^2 + (\alpha y - \gamma w + \delta x - \beta z)^2 + (\alpha z - \delta w + \beta y - \gamma x)^2$; or, for the sake of abridging this expression, ${}''AA = w'^2 + x'^2 + y'^2 + z'^2$. If now ${}''A = 1$, $A = w'^2 + x'^2 + y'^2 + z'^2$, and the proposition is demonstrated: but if ${}''A > 1$, we must proceed in the same manner to obtain a new product ${}'''AA$, expressed by four squares, in which we shall have ${}'''A \angle {}''A$; continuing thus the decreasing series of integral numbers A, ${}'A$, ${}''A$, ${}'''A$, &c. we must necessarily arrive at a term ${}^{m}A$, equal to unity, in which case the prime number A will be expressed by the sum of four squares. ………………………………… *Q. E. D.*

THEOREM.

Every integral number whatever, is either a square, or the sum of *two*, *three*, or *four* squares.

This is an immediate consequence of the last proposition and of the formulæ at Prop. 3; for every number is either a prime, or produced by the multiplication of prime factors, and since every prime number is of the form $(w^2 + x^2 + y^2 + z^2)$, and the product of two or more such formulæ is still of the same form; it necessarily follows, that every integral number whatever may be represented by $(w^2 + x^2 + y^2 + z^2)$; where it remains to be observed, that no limitation in the course of the demonstration has been made, that can prevent any *one*, or *more* of those squares from becoming zero; therefore every integral number whatever, is either a square, or the sum of *two*, *three*, or *four* squares. .... Q. E. D.

Cor. All that has been proved for integral numbers, is equally true for fractional ones; for every fraction may be reduced to an equivalent fraction having a square denominator; therefore every fraction is of the form

$$\frac{w^2 + x^2 + y^2 + z^2}{m^2} = \frac{w^2}{m^2} + \frac{x^2}{m^2} + \frac{y^2}{m^2} + \frac{z^2}{m^2};$$

this curious property extends, therefore, to every rational number whatever.

ARTICLE XX.

Two letters from Mr. KNIGHT, *containing a demonstration of the Binomial Theorem.*

LETTER I.

To the Editor of the Mathematical Repository.

SIR,

There is a very simple relation between the coefficients of two successive powers of a binomial: it furnishes the following demonstration of the binomial theorem in the case of whole positive powers; which, however, I do not pretend to be of much merit.

$$(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4 \quad \ldots\ldots\ldots \quad (\alpha)$$
$$(1 + x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 \quad (\beta)$$

(α) multiplied by 5 gives

$$5(1+x)^4 = 5 + 20x + 30x^2 + 20x^3 + 5x^4 \ldots\ldots (\gamma)$$

If we put $5 = m$, and represent (β) thus,

$$(1+x)^m = 1 + {'A}x + {''A}x^2 + {'''A}x^3 + {''''A}x^4 + {'''''A}x^5 \quad (\delta)$$

it is plain by comparing the coefficients of (β) and (γ), that the latter will become

$$m(1+x)^{m-1} = {'A} + 2{''A}x + 3{'''A}x^2 + 4{''''A}x^3 + \&c. \ldots\ldots (\epsilon).$$

But here let it be particularly observed, that m is not intended to represent any number generally, but only the particular number 5. Multiply (ϵ) by $1 + x$ and there arises

$$m(1+x)^m = {'A} + \left.\begin{array}{l} 2{''A} \\ + {'A} \end{array}\right\} x + \left.\begin{array}{l} 3{'''A} \\ + 2{''A} \end{array}\right\} x^2 + \left.\begin{array}{l} 4{''''A} \\ + 3{'''A} \end{array}\right\} x^3 +$$

which equation being compared with (δ) gives

$${'A} = m,\ {''A} = \frac{m(m-1)}{2},\ {'''A} = \frac{m(m-1)(m-2)}{2 \cdot 3}, \&c.$$

wherefore in the particular case in which m represents the number 5, we have

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2}x^2 + \&c. \ldots\ldots (\theta).$$

It remains to be shewn, that if this equation be true of one value of m, it is so with respect to every other also.

For this purpose multiply (θ) by $1 + x$ and there will arise

$$(1+x)^{m+1} = 1 + (m+1)x + \frac{(m+1)m}{2}x^2 + \&c.$$

which completes the demonstration.

I am Sir,

Your obedient Servant,

THOS. KNIGHT.

Papcastle,
Oct. 28, 1808.

LETTER II.

To the Editor of the Mathematical Repository.

SIR,

Having before sent you a demonstration of the binomial theorem, in the case of a whole positive exponent, it may not be amiss to shew briefly, that the form is exactly the same in that of a positive or negative fraction. I ought, however, to mention, that the following demonstration, though simpler, does not differ *in principle* from one given by *Garnier*, in his edition of *Clairaut's Algebra*.

Suppose that when m is a whole number $(1+x)^m = A + {}'Am + {}''Am^2 +$; because $\{(1+x)^m\}^n = (1+x)^{mn}$ we shall have, if n is a whole positive number, and consequently mn,

$$(A + {}'Am + {}''Am^2 +)^n = A + {}'Amn + {}''Am^2n^2 + \ldots\ldots (\alpha).$$

Now this equation must plainly be identical, independently of the values of m and n; let then $m = \frac{s}{n}$, and it becomes

$$(A + {}'A\frac{s}{n} + {}''A\frac{s^2}{n^2} +)^n = A + {}'As + {}''As^2 + = (1+x)^s,$$

because s is a whole positive number.

We have then $A + {}'A\frac{s}{n} + {}''A\frac{s^2}{n^2} + = (1+x)^{\frac{s}{n}}$; whence it appears that by changing, in the expansion of $(1 + x)^m$, m into $\frac{s}{n}$, we have that of $(1 + x)^{\frac{s}{n}}$.

In equation (α) change n into $-n$, and m into $-\frac{s}{n}$, and it becomes

$$(A - {}'A\frac{s}{n} + {}''A\frac{s^2}{n^2} -)^{-n} = A + {}'As + {}''As^2 + = (1+x)^s.$$

Whence $A - {}'A\frac{s}{n} + {}''A\frac{s^2}{n^2} - = (1 + x)^{-\frac{s}{n}}$: so that, by changing m into $-\frac{s}{n}$, the same expansion is adapted to the case of a negative fraction.

Therefore, in all cases,

$$(1 + x)^m = 1 + mx + \frac{m(m-1)}{1.2}x^2 + \&c.$$

Papcastle,
Decemb. 7, 1808.

THOMAS KNIGHT.

ARTICLE XXI.

A Dynamical Principle. By Mr. THOMAS BAZLEY.

To the Editor of the Mathematical Repository.

SIR,

The following dynamical principle suggested itself to me some time ago, in considering a particular problem. The principle itself is an immediate consequence of the third law of motion; and, notwithstanding its simplicity, it has remained hitherto unnoticed by any writer that I am acquainted with. I have shewn its application to a few known examples, which I think will sufficiently evince the utility and generality of this principle. If it will not occupy too much room in your valuable work, I shall be obliged by its insertion. I am respectfully your's, &c.

Bolton, Oct. 24, 1808, THOMAS BAZLEY.

If the motion of a body P be any how opposed by the actions of several others connected with it, and the motive forces of these be reduced to the direction of P, and subtracted from its motive force, the remainder is the motive force with which P is urged in consequence of its connexion.

In what follows, I shall confine myself to the determination of the accelerative force; because, when that is known, the circumstances of motion may be determined by the resolution of known equations, (as $f\dot{x} = v\dot{v}$, $\dot{t} = \dot{x} \div v$, &c.) and consequently the difficulties that occur are purely mathematical.

Ex. 1. Let the bodies p and q be connected by a thread of a given length, and suppose q to be placed on a smooth horizontal plane; and the thread passing perpendicularly over the edge of the plane, let p hang freely: query the force that accelerates p?

If $g = 16\frac{1}{12}$ feet, then will $2g$ be the measure of the accelerative force of gravity: let x denote the space passed over by p or q, t the time of motion, and v the velocity acquired in that time. Then the motive force of p is $2gp$, and that of q arising from its inertia is $(\dot{v} \div \dot{t})\,q$ as is well known; therefore, by the principle above, the motive force whereby p is now urged, is $2gp - (\dot{v} \div \dot{t})\,q = (\dot{v} \div \dot{t})\,p$ by the known principles, therefore $\frac{\dot{v}}{\dot{t}}(p + q) = 2gp$, or $\frac{\dot{v}}{\dot{t}} = \frac{2gp}{p+q} =$ the accelerative force; and therefore the circumstances of motion are given by the laws of constant forces. If instead of p we had taken some other moving force, destitute of inertia, which we shall denote by m; we should have found $m - (\dot{v} \div \dot{t})\, q = (\dot{v} \div \dot{t}) \times 0$, because the mass of m is 0, therefore $\frac{\dot{v}}{\dot{t}} \cdot q = m$, or $\frac{\dot{v}}{\dot{t}} = \frac{m}{q} =$ the accelerative force.

Ex. 2. The other notation remaining, let q be placed on an inclined plane, s denoting the sine of inclination; then the motive force of p being $2gp$, the motive force of q arising from inertia is $(\dot{v} \div \dot{t})\, q$, and that arising from its gravity is $2g \times s \times q$, and both of these forces resist the motion of p: consequently by the principle

$$2g\,(p - sq) - \frac{\dot{v}}{\dot{t}}\, q = \frac{\dot{v}}{\dot{t}}\, p, \text{ and therefore } \frac{\dot{v}}{\dot{t}} = 2g\left(\frac{p - sq}{p + q}\right) =$$

the accelerative force.

If $s = 1$, it becomes the case of two bodies connected by a string passing over the edge of the plane when vertical, or if you please, over a pully; and then $\frac{\dot{v}}{\dot{t}} = \frac{p - q}{p + q} \times 2g$.

Ex. 3. Suppose p to be applied by a thread to a wheel, to elevate q applied by a thread to the axle; and let v denote the velocity of p, b the radius of the wheel, and a that of the axle. Then will $2gp$ be the motive force of p, and $2gq =$ the motive force of q at a arising from gravity; but $b : a :: 2gq : 2gqa \div b$ = motive force of q at b arising from gravity. Again $b : a :: v : av \div b$ = velocity of q, and consequently the motive force at a arising from its inertia is $\dot{v}a \div \dot{t}b$, and $b : a :: \dot{v}a \div \dot{t}b : \dot{v}a^2 \div \dot{t}b^2$ = force of its inertia acting at b: Hence the force with which p descends in consequence of its connexion is

$$2g\,(p - (a \div b)\, q) - (a^2\dot{v} \div b^2\dot{t})\, q = (\dot{v} \div \dot{t})\, p; \text{ therefore}$$

$$\frac{\dot{v}}{\dot{t}}\left(\frac{pb^2 + qa^2}{b^2}\right) = 2g\,\left(p - \frac{a}{b}q\right), \text{ or } \frac{\dot{v}}{\dot{t}} = 2g \cdot \frac{p - \frac{a}{b}q}{p + \frac{a^2}{b^2}q} = \text{the}$$

accelerative force.

If we consider p as applied at the distance b to communicate motion, in a horizontal plane, to q placed at the distance a from the same centre: then the gravity of q would not affect the motion of p and we should have $\frac{\dot{v}}{\dot{t}} = 2gp \div (p + (a^2 \div b^2)\, q)$; and if we further suppose instead of p, some other power m void of inertia, we shall have $\frac{\dot{v}}{\dot{t}} = \frac{m}{(a^2 \div b^2)\, q} = \frac{mb^2}{qa^2}$.

Hence appears the facility with which the properties of circular motion are deducible from our principle.

Ex. 4. The curious problem at page 131 of Simpson's Miscellaneous Tracts (of which an elegant solution is given by Mr. Ivory, at page 22, part II. vol. I. of the Mathematical Repository, New Series) is easily solved by this principle.

PROBLEM. Fig. 26, Pl. D.

Suppose that a thread AC *n* CA, having two equal weights A, A suspended at the ends thereof, is hung over two tacks C, C, in the same horizontal line; and that to the middle point of the thread (*n*), equally distant from the tacks, another given weight B is fixed, which is permitted to descend by its own gravity, so as to cause the other two weights, at the same time, to ascend: it is proposed to find the law of the velocity by which the said weights ascend and descend; abstracting from the resistance of the air, the weight of the thread, and the friction of the tacks.

Let v denote the velocity of B (measured by the distance that might be uniformly gone over in one second of time), and let b ($= 32\frac{1}{6}$ feet) = the measure of the velocity, which gravity can generate in a falling body in *one second*; putting $\mathrm{CE} = a$, $\mathrm{E}n = x$, $\mathrm{C}n = y$: then we shall have the motive force of $\mathrm{B} = b x \mathrm{B}$ = force of B to elevate both the weights A, A. The gravity of A is $b\mathrm{A}$, which reduced to the direction $n\mathrm{E}$ is $b\mathrm{A}\,(x \div y)$, and consequently, since both weights act against B, the loss of force from their gravity will be $2b\mathrm{A}\,(x \div y)$. Mr. S. shews that the velocity wherewith A ascends is $(v \div y)\sqrt{(y^2 - a^2)}$ whose fluxion is $((y^2 - a^2)\,y\dot{v} + a^2 v\dot{y}) \div y^2 x$, and therefore the resistance of A in direction ACn arising from inertia is $((y^2 - a^2)\,y\dot{v} + a^2 v\dot{y}) \times \mathrm{A} \div y^2 x\dot{t}$; and this reduced to the direction $n\mathrm{E}$ is $((y^2 - a^2)\,y\dot{v} + a^2 v\dot{y}) \times \mathrm{A} \div y^3\dot{t}$; and for both weights it is $\dfrac{(y^2 - a^2)\,y\dot{v} + a^2 v\dot{y}}{y^3\dot{t}} \times 2\mathrm{A}$.

Therefore by the principle

$$b\left(\mathrm{B} - \frac{2\mathrm{A}x}{y}\right) - \frac{(y^2 - a^2)\,y\dot{v} + a^2 v\dot{y}}{y^3\dot{t}} \times 2\mathrm{A} = \frac{\dot{v}}{\dot{t}}\,\mathrm{B},$$

or, since $v\dot{t} = \dot{x}$ and $\dot{v} \div \dot{t} = v\dot{v} \div \dot{x}$, by reduction we have

$$b\dot{x} - b \cdot \frac{2\mathrm{A}x\dot{x}}{\mathrm{B}y} - 2\mathrm{A} \times \frac{(y^2 - a^2)\,yv\dot{v} + a^2 v^2\dot{y}}{\mathrm{B}y^3} = v\dot{v},$$ which

is the very equation deduced by Mr. S. from a process quite different. Hence substituting $y\dot{y}$ for $x\dot{x}$ &c. his conclusions will follow.

Ex. 5. The last application we shall make of this principle shall be to the general problem at page 285, of Mr. Atwood's Treatise on Motion.

PROBLEM. Fig. 27, Pl. D.

Let AFC, BGC, be two curves, the planes of which are vertical; and let a line, ACB, be stretched over a fixed pully, C, by two given weights A and B, of which B preponderates against A, and descends along the curve CGB; it is required to assign the velocity of B, when it has descended through a given perpendicular altitude, and A has ascended through a given altitude in the same time.

Let BL be the perpendicular altitude through which B has descended, and let AH be the perpendicular altitude through which A has ascended in the same time, and let $BL = y$, $AH = q$. Suppose the weight B to describe in its descent the evanescent arc *b*B, and during the same time let the other weight A rise through the arc *a*A. Through *b* and A draw *bo* and A*l* parallel to the horizon; also, through *b* draw *bn* perpendicular to CB, and A*m* perpendicular to C*a*, and let $CB = x$, $Bn = \dot x = Am$, $Bo = \dot y$, $Bb = \dot s$, $Aa = \dot r$, and $al = \dot q$. We find, as Mr, Atwood has done, the motive force of A arising from gravity and reduced to the direction of $B = A\dot q \div \dot s$; and the resistance from A's inertia reduced to the same direction is $u\dot u \div \dot s$, theref. by the principle $\frac{B\dot y}{\dot s} - \frac{A\dot q}{\dot s} - \frac{Au\dot u}{\dot s} = \frac{Bv\dot v}{\dot s}$, where u = vel. of A and v = vel. of B.

But $\dot s : \dot r :: v : u = (\dot r \div \dot s)v$, and (since $v\dot t = \dot s$ and $\dot v\dot t = \ddot s$, &c. therefore $\ddot s : \ddot r :: \dot v : \dot u = (\ddot r \div \ddot s)\dot v$; whence $u\dot u = (\dot r\ddot r \div \dot s\ddot s)v\dot v$, therefore $B\dot y - A\dot q = Au\dot u + Bv\dot v = (A\dot r\ddot r + B\dot s\ddot s)v\dot v \div \dot s\ddot s$: consequently the accelerating force of $B = \frac{v\dot v}{\dot s} = \frac{(B\dot y - A\dot q)\ddot s}{A\dot r\ddot r + B\dot s\ddot s}$ which is the identical expression found by Mr. Atwood.

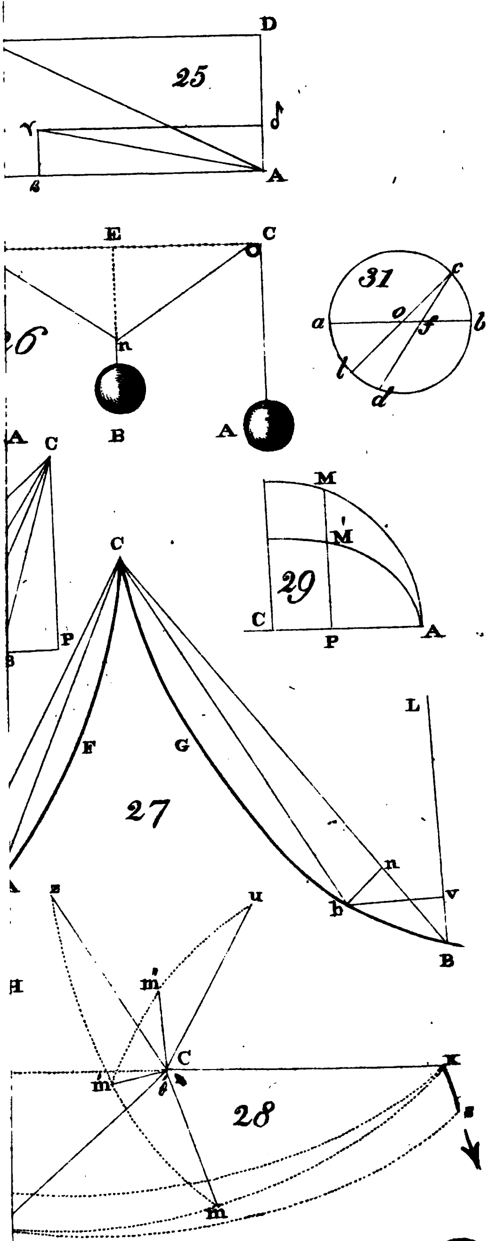
End of the Second Part of the Second Volume.

ERRATUM.

Part 2nd. page 53, line 11 from the bottom, for "infinite" read "finite".

W. Glendinning, Printer, Hatton Garden, London.

Vol. II. Plate D, Fig. 25 to 31.



UNIV. OF MICH.

NEW SERIES

OF THE

MATHEMATICAL REPOSITORY.

VOL. II. PART III.

MATHEMATICAL MEMOIRS, EXTRACTED FROM WORKS OF EMINENCE.

ARTICLE I.

A MEMOIR ON ELLIPTIC TRANSCENDENTALS.

Containing eaſy methods of comparing and valuing theſe Tranſcendentals, which include elliptic arches, and which frequently occur in the application of the Intègral Calculus[*].

(Read to the *Ci-devant* Academy of Sciences, in April, 1792.)

By ADRIEN MARIE LE GENDRE, *Member of the National Inſtitute of France, and* F. R. S. L.

Many integrals may be determined ſolely by the aid of arches of a circle and logarithms, which are the moſt ſimple tranſcendentals; but to extend the application of the integral calculus, recourſe muſt be had to tranſcendentals of a more complex form. It is by examining with care the nature and properties of theſe, at leaſt ſuch as are of more frequent uſe, that we can ſimplify conſiderably the reſults of the theory, and render it more convenient in practice.

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[*] LACROIX observes, in a note to his valuable Treatise on the Integral Calculus that this interesting Memoir, which makes no part of any academical collection, was become very rare, there having been but few copies of it orignally printed. It is inserted here at the request of several eminent Mathematicians.

Next to arches of a circle and logarithms, elliptic arches are one of the moſt ſimple tranſcendentals, and they might be made in ſome meaſure a new inſtrument of calculus, if their properties were once commonly known, and we poſſeſſed eaſy methods of eſtimating them accurately. Upon this ſubject, we have propoſed ſome new views, in the Memoirs of the Academy of Sciences of Paris, for the year 1786.

But if elliptic arches add greatly to the means of analyſis, more eſpecially when it is conſidered that hyperbolic arches may be eaſily deduced from them, they are, however, inadequate to ſome inquiries which are in ſome reſpects but little complicated, ſuch as the ſurface of an oblique cone, the motion of rotation of a body which is not impelled by any accelerating force, &c. Theſe and innumerable others depend upon the integral $\int \frac{P\,dx}{R}$, in which P is a rational function of x, and R a radical of the form $\sqrt{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4)}$, α, β, γ, &c. being conſtant. Now a careful examination of the nature of this integral ſhews, that it preſents three diſtinct ſpecies of tranſcendentals. The firſt and ſecond may be expreſſed by elliptic arches, the third is more complex, but it has ſuch an analogy with the two others, that all three may be conſidered as conſtituting but one and the ſame order of tranſcendentals, the next to arches of a circle and logarithms.

As the two firſt ſpecies are expreſſible by arches of ellipſes, they might have been included in one, however we have thought proper to diſtinguiſh them, obſerving that what we call the firſt ſpecies may be determined by the ſecond, but not the ſecond by the firſt: hence it appears that the firſt ſpecies conſidered analytically is more ſimple in its nature than the other; and therefore, that elliptic arches are not the moſt ſimple tranſcendental that occurs after arches of a circle and logarithms. Upon the whole, the three ſpecies of tranſcendentals, which we are conſidering, are ſo related to each other, that we have judged it proper to give them the common name of *Elliptic Tranſcendentals*.

We propoſe in this memoir to explain the nature and properties of this order of tranſcendentals, ſo as to render its uſe eaſy in the application of the integral calculus. Some of the methods and reſults which we ſhall explain are already known to geometers; we have brought into one point of view all that has hitherto been publiſhed relating to this theory, but we have, at the ſame time, endeavoured to give it a greater degree of perfection.

A general idea of the different kinds of tranſcendentals, contained in the integral formula $\int \frac{Pdx}{R}$.

1. We ſhall put P for any rational function whatever of x; and R for the radical $\sqrt{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4)}$: this radical remaining the ſame, an infinity of different values may be given to P, but there will not thence reſult an infinity of different tranſcendentals; for we ſhall ſhew that the integral $\int \frac{Pdx}{R}$ may always, by partial integrations, be reduced to an algebraic part *plus* a certain number of tranſcendentals, which are all of the ſame form and nature.

Firſt, let us ſuppoſe that P is an integer function of x, ſuch, that $P = A + Bx + Cx^2 + \ldots\ldots + Kx^k$. If, in order to abridge, we repreſent the integral $\int \frac{x^m dx}{R}$ by Π^m, it is evident that $\int \frac{Pdx}{R} = A\Pi^0 + B\Pi^1 + C\Pi^2 + \ldots . K\Pi^k$; now, by taking the differential of the quantity $x^{m-3}R$, and then returning from the differential to the integral, we get this formula

$$x^{m-3}R = (m-3)\alpha\Pi^{m-4} + (m-\tfrac{5}{2})\beta\Pi^{m-3} + (m-2)\gamma\Pi^{m-2} + (m-\tfrac{3}{2})\delta\Pi^{m-1} + (m-1)\varepsilon\Pi^m;$$

hence it follows, that Π^m, Π^{m-1}, &c. may be reduced ſucceſſively to lower degrees; and that the reduction may be repeated till the quantities be all of a lower degree than Π^3, for the reduction of Π^3 is alſo poſſible, becauſe, that in the caſe of $m = 3$, the term which would ſeem to contain Π^{-1} vaniſhes. Therefore, P being any integer function of x, the integral $\int \frac{Pdx}{R}$ may always be reduced to an algebraic quantity, together with a tranſcendental, which in every caſe will be of this form

$$\int (A + Bx + Cx^2)\frac{dx}{R}.$$

2. Let us now conſider the formula in its moſt general form, and let P be any rational function whatever of x; we proceed

ceed as in the integration of the rational fraction Pdx, and in the firſt place we ſeperate the integer part of it, which may be treated as has been already explained. We next decompoſe the remaining part into ſeveral partial fractions, according to the number of factors in the denominator; theſe conſtitute ſo many terms of the whole integral, and each term may be repreſented by the general expreſſion $N\int\frac{dx}{(1+nx)^k R}$. But we ſhall immediately ſhew that it is eaſy to reduce all the terms of this form to ſimilar terms in which $k = 1$. And in the firſt place, we obſerve that if the denominator, inſtead of being complex, were ſimply x^k, the integral $\int\frac{dx}{x^k R}$, and all ſimilar to it in which $k > 1$, might be determined by means of the integral $\int\left(\frac{A}{x} + B + Cx + Dx^2\right)\frac{dx}{R}$; for this purpoſe, it would only be neceſſary to give negative values to $m - 4$, in the formula of article 1.

Let us now return to the general term $\int\frac{dx}{(1+nx)^k R}$, which we ſhall call Γ^k. If we put $1 + nx = \omega$, and take the co-efficients, α', β', &c. ſo as to have the identical equation

$$\alpha + \beta\left(\frac{\omega - 1}{n}\right) + \gamma\left(\frac{\omega - 1}{n}\right)^2 + \delta\left(\frac{\omega - 1}{n}\right)^3 + \varepsilon\left(\frac{\omega - 1}{n}\right)^4$$
$$= \alpha' + \beta'\omega + \gamma'\omega^2 + \delta'\omega^3 + \varepsilon'\omega^4,$$

we ſhall find by the differentiation of the quantity $\omega^{-k+1}R$ this general formula

$$\frac{-R}{n(1+nx)^{k-1}} = (k-1)\alpha'\Gamma^{k} + (k - \tfrac{3}{2})\beta'\Gamma^{k-1} + (k-2)\gamma'\Gamma^{k-2}$$
$$+ (k - \tfrac{5}{2})\delta'\Gamma^{k-3} + (k-3)\varepsilon'\Gamma^{k-4}.$$

Hence it follows, that the quantities Γ^2, Γ^3, &c. may be determined by means of Γ^1, Γ^0, Γ^{-1}, Γ^{-2}; but $\Gamma^0 = \int\frac{dx}{R}$, $\Gamma^{-1} = \int(1 + nx)\frac{dx}{R}$, $\Gamma^{-2} = \int(1 + nx)^2\frac{dx}{R}$. Therefore, in general the formula $\int\frac{dx}{(1+nx)^k R}$, and all others ſimilar to it, in which k is greater than unity, are reducible to an algebraic part, together with an integral of this form

$$\int\left(\frac{A}{1+nx}+B+Cx+Dx^2\right)\frac{dx}{R}.$$

3. Hence we conclude, that P being put for any rational function whatever of x, the integral $\int\frac{P\,dx}{R}$ is always reſolvable into three principal parts, the firſt is algebraic, the ſecond has this form $\int(A+Bx+Cx^2)\frac{dx}{R}$, and the third contains one or more terms of the form $N\int\frac{dx}{(1+nx)R}$, where the coefficient n may be either real or imaginary.

It appears by this analyſis that the number of tranſcendentals contained in the formula $\int\frac{P\,dx}{R}$ is very limited, for there are only two principal ſpecies, the one having the form $\int(A+Bx+Cx^2)\frac{dx}{R}$, and the other the form $\int\frac{dx}{(1+nx)R}$. I even obſerve, that as often as n is real, the ſecond ſpecies is contained in the firſt, and may be immediately reduced to it by making $1+nx=\frac{1}{z}$.

Theſe general reductions being once underſtood we proceed in a different manner to attain a more preciſe knowledge of the ſame tranſcendentals.

Of the way in which the odd powers of the variable quantity under the radical may be made to diſappear.

4. The radical being always $\sqrt{(\alpha+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4)}$, let us ſuppoſe that the quantity under the ſign is reſolved into two real factors $\zeta+2\eta x+\theta x^2$, $\lambda+2\mu x+\nu x^2$. Theſe factors ought to have the ſame ſign that the radical may be real; therefore we may ſuppoſe that

$$\zeta+2\eta x+\theta x^2=(\lambda+2\mu x+\nu x^2)\,y^2.$$

Hence we find

$$x=\frac{\mu y^2-\eta+\sqrt{\left\{(\mu y^2-\eta)^2+(\lambda y^2-\zeta)(\theta-\nu y^2)\right\}}}{\theta-\nu y^2};$$

and, if in order to abridge, Y is put for the radical of this expreſſion, we have $\frac{dx}{R} = \frac{dy}{Y}$. The value of x being ſubſtituted in P, it appears that the differential $\frac{Pdx}{R}$, or $\frac{Pdy}{Y}$ may be always reſolved into two parts, the one rational and integrable by the common rules, and the other affected by the radical Y, but ſuch, that only the even powers of y are found either under the radical or without it. Thus the integration of the formula $\frac{Pdx}{R}$ is always reducible to that of a formula of the ſame kind, but in which P and R contain only the even power of x.

5. This method is general; however, as the value of x which ought to be ſubſtituted in P is a little complex, it may not be uſeleſs to ſhew that the ſame end may be obtained by a much more ſimple ſubſtitution, which conſiſts in putting $x = \frac{p + qy}{1 + y}$, p and q being two indeterminate conſtant quantities.

Reſuming now the factors $\zeta + 2\eta x + \theta x^2$, $\lambda + 2\mu x + \nu x^2$, and ſubſtituting in each of them the value of x, we may leave out of conſideration the common denominator, which goes out of the radical, then, that the odd powers of y may diſappear in the numerators, it is neceſſary that

$$\zeta + \eta\,(p + q) + \theta pq = 0.$$
$$\lambda + \mu\,(p + q) + \nu pq = 0.$$

Theſe two equations give rational values of $p + q$ and pq, but that p and q may be real, it is neceſſary that $(p + q)^2 - 4pq$, or $(p - q)^2$ be poſitive. Now we may diſtinguiſh two caſes.

1ſt. If the factors of the quantity $\alpha + \beta x + \gamma x^2 +$ &c. are not all real, let it be ſuppoſed that $\lambda + 2\mu x + \nu x^2$ contains two of them which are imaginary, and conſequently, that $\lambda\nu > \mu^2$. But the ſecond equation containing p and q, gives

$$(p - q)^2 = \left(\frac{\lambda + \nu pq}{\mu} - \frac{2\mu}{\nu}\right)^2 + \frac{4\lambda\nu - 4\mu^2}{\nu^2};$$

therefore, in the firſt caſe, the ſecond member of the equation is poſitive, and the values of p and q are real.

2d. If all the factors of the quantity $\alpha + \beta x + \gamma x^2 +$ &c. be real, and if that in reſolving it into two factors of the ſecond degree, as has been done, the values of p and q are not real, then it is to be obſerved that there are two other ways in which a quantity of the fourth degree may be reſolved into two factors of the ſecond, and we may be aſſured that theſe two new combinations will give real values of p and q. This we ſhall now demonſtrate.

Let a, b, c, d, be the four values of x which are found by making $R = 0$, the equations which determine p and q may be written thus,

$$ab - \cancel{\tfrac{1}{2}}(a + b)(p + q) + pq = 0$$
$$cd - \cancel{\tfrac{1}{2}}(c + d)(p + q) + pq = 0;$$

Hence we get

$$p + q = \frac{ab - cd}{a + b - c - d},$$

$$\left(\frac{p - q}{2}\right)^2 = \frac{(a - c)(a - d)(b - c)(b - d)}{(a + b - c - d)^2}.$$

But it is always in our power to render the denominator of this laſt quantity poſitive; for ſuppoſe that a, b, c, d, are written in the order of their magnitudes, the negative quantities coming after the poſitive, then the differences $a - b$, $a - c$, &c. ſhall all be poſitive, and $p - q$ real. We ſhall alſo have $p - q$ real, if for a and b we take the greateſt and leaſt of the four roots a, b, c, d. Finally, the third combination gives $p - q$ imaginary.

Therefore, by the ſubſtitution, $x = \frac{p + qy}{1 + y}$, it is always poſſible to cauſe the odd powers of the variable quantity under the radical to diſappear, and at the ſame time the two factors under the new radical are real and of the form $f + gy^2$, which is a new and important advantage, and which cannot be obtained by the firſt ſubſtitution.

Obſerve that there are two particular caſes to be examined.

1ſt. When one of the roots a and b is equal to one of the roots c and d.

2d. When $a + b = c + d$. In the firſt caſe the radical R becomes more ſimple, and the integral depends only upon arches of a circle and logarithms. With reſpect to the ſecond caſe, a ſimple permutation is ſufficient to prevent $a + b$ from being equal to $c + d$, but this circumſtance may be employed to facilitate the transformation. For the two factors under the radical having then the form $\lambda + \nu(x^2 + 2mx)$, $\zeta + \theta(x^2 + 2mx)$, it is evident that if we put $x + m = y$, the uneven powers of the variable quantity diſappear.

Reduction

Reduction of the differential $\frac{Pdx}{R}$ *to the form* $\frac{Qd\phi}{\sqrt{(1-c^2\,\text{ſin.}^2\,\phi)}}$.

6. Since by a previous preparation we can eliminate the odd powers of the variable quantity under the radical, we ſhall in future put $R = \sqrt{(\alpha + \beta x^2 + \gamma x^4)}$, we ſhall alſo ſuppoſe that P is an even function of x, for whatever be the rational function P, we can always make $P = M + Nx$, M and N being even functions of x; now by making $x^2 = y$, the part $\frac{Nxdx}{R}$ is reducible to the common rules, and thus all the difficulty is thrown upon the integration of the formula $\frac{Mdx}{R}$, in which M is an even function of x.

This being underſtood, we proceed to prove, by enumerating the different caſes, that the differential $\frac{dx}{R}$ may always be reduced to the form $\frac{md\phi}{\sqrt{(1-c^2\,\text{ſin.}^2\,\phi)}}$, c being leſs than unity.

FIRST CASE. Suppoſe that the factors of $\alpha + \beta x^2 + \gamma x^4$ are imaginary; let this quantity be expreſſed by $\lambda^2 + 2\lambda\mu x^2 \cos.\,\theta + \mu^2 x^4$, λ and μ being poſitive, and $\cos.\,\theta$ either poſitive or negative. We make $x = \sqrt{\left(\frac{\lambda}{\mu}\right)} \tan.\,\frac{1}{2}\phi$, then, taking $c = \text{ſin.}\,\frac{1}{2}\theta$, we have the transformed equation

$$\frac{dx}{R} = \frac{1}{2\sqrt{\lambda\mu}} \cdot \frac{d\phi}{\sqrt{(1-c^2\,\text{ſin.}^2\,\phi)}}.$$

SECOND CASE. Let $\alpha + \beta x^2 + \gamma x^4 = m^2(1 + p^2x^2)(1 - q^2x^2)$, the limit of x being $\frac{1}{q}$; by putting $x = \frac{1}{q}\cos.\,\phi$ and $\frac{p^2}{p^2+q^2} = c^2$ we get

$$\frac{dx}{R} = \frac{c}{mp} \cdot \frac{-d\phi}{\sqrt{(1-c^2\,\text{ſin.}^2\,\phi)}}.$$

If we wiſh the transformed equation to be poſitive, we muſt make $\cot.\,\phi = \sqrt{(1-c^2)}\tan.\,\psi$, and hence we have immediately $x^2 = \frac{\text{ſin.}^2\,\psi}{q^2 + p^2\cos.^2\,\psi}$, and the transformed equation

$$\frac{dx}{R} = \frac{c}{mp} \cdot \frac{d\psi}{\sqrt{(1-c^2\,\text{ſin.}^2\,\psi)}}.$$

THIRD CASE. Let $\alpha+\beta x^2+\gamma x^4=m^2(1+p^2x^2)(x^2-q^2)$; we put $x=\frac{q}{\text{coſ.}\,\varphi}$, $\frac{1}{1+p^2q^2}=c^2$, and hence we get

$$\frac{dx}{R}=\frac{c}{m}\cdot\frac{d\varphi}{\surd(1-c^2\,\text{ſin.}^2\,\varphi)}.$$

FOURTH CASE. Let $\alpha+\beta x^2+\gamma x^4=m^2(1+p^2x^2)\times(1+q^2x^2)$, and let $p>q$, by putting $x=\frac{\text{tan.}\,\varphi}{p}$, and $\frac{p^2-q^2}{p^2}=c^2$, we have

$$\frac{dx}{R}=\frac{1}{mp}\cdot\frac{d\varphi}{\surd(1-c^2\,\text{ſin.}^2\,\varphi)}.$$

FIFTH CASE. Let $\alpha+\beta x^2+\gamma x^4=m^2(1-p^2x^2)\times(1-q^2x^2)$, and let $p>q$, then, putting $px=\text{ſin.}\,\varphi$, and $\frac{q}{p}=c$, the transformed equation becomes

$$\frac{dx}{R}=\frac{1}{mp}\cdot\frac{d\varphi}{\surd(1-c^2\text{ſin.}^2\varphi)}.$$

This formula is applicable from $x=0$ to $x=\frac{1}{p}$; the radical R is imaginary from $x=\frac{1}{p}$ to $x=\frac{1}{q}$; but it becomes again real from $x=\frac{1}{q}$ to $x=\infty$. In the laſt caſe it is neceſſary to write $R^2=m^2(p^2x^2-1)(q^2x^2-1)$; then, making $qx=\frac{1}{\text{ſin.}\,\varphi}$, and $\frac{q}{p}=c$, the transformed equation becomes $\frac{-d\varphi}{mp\surd(1-c^2\,\text{ſin.}^2\,\varphi)}$. If we wiſh to have it poſitive, we muſt make, as before, $\text{cot.}\,\varphi=\surd(1-c^2)\,\text{tan.}\,\psi$, which immediately gives $x^2=\frac{1-c^2\,\text{ſin.}^2\,\psi}{q^2\,\text{coſ.}^2\,\psi}$, and the transformed equation is

$$\frac{dx}{R}=\frac{1}{mp}\cdot\frac{d\psi}{\surd(1-c^2\,\text{ſin.}\,\psi)},$$

a formula perfectly ſimilar to the firſt, from which it follows that the integral of $\frac{dx}{R}$, in the caſe of $x>\frac{1}{q}$ may always be found from the ſame integral taken upon the ſuppoſition of $x<\frac{1}{p}$.

Sixth Case. Laſtly. Let $\alpha+\beta x^2+\gamma x^4 = m^2(x^2-q^2)\times(p^2-x^2)$; then x muſt be contained between p and q. Let $p > q$, and aſſume $x = \frac{q}{\text{coſ.}\,\psi}$; then we have immediately

$$\frac{dx}{\text{R}} = \frac{d\psi}{m\sqrt{(p^2-q^2-p^2\,\text{ſin.}^2\,\psi)}}.$$

In this formula the angle ψ has a limit; but that we may introduce into it one which is indefinite, let $\text{ſin.}\,\psi = c\,\text{ſin.}\,\phi$, and $c^2 = \frac{p^2-q^2}{p^2}$, thus the formula is transformed to

$$\frac{dx}{\text{R}} = \frac{1}{mp}\cdot\frac{d\phi}{\sqrt{(1-c^2\,\text{ſin.}^2\,\phi)}},$$

to which we might have come directly by making $x^2 = \frac{q^2}{1-c^2\,\text{ſin.}^2\,\phi}$.

7. It may now be remarked that, ſetting aſide the firſt caſe, the values which muſt be given to x^2, ſo as to produce the required reduction, have always the form $\frac{\text{A}+\text{B}\,\text{ſin.}^2\,\phi}{\text{C}+\text{D}\,\text{ſin.}^2\,\phi}$, where the coefficients are conſtant. Therefore, it is only neceſſary to ſubſtitute this value of x^2 in P, and the formula $\int\frac{\text{P}dx}{\text{R}}$ ſhall be transformed into another $\int\frac{\text{Q}d\phi}{\sqrt{(1-c^2\,\text{ſin.}^2\,\phi)}}$, in which c is leſs than unity, and Q is a rational and even function of $\text{ſin.}\,\phi$, which contains $\text{ſin.}\,\phi$ to the ſame degree that P contains x.

We ſet aſide the firſt caſe, becauſe of the form of the value of x being a little different, and alſo, becauſe that by following the method of article 5, this caſe is avoided, ſeeing that we always get real factors. But in another article we ſhall return to the caſe of imaginary factors.

Developement of the formula $\int\frac{\text{Q}d\phi}{\sqrt{(1-c^2\,\text{ſin.}^2\phi)}}$.

8. We ſhall henceforth denote the radical $\sqrt{(1-c^2\,\text{ſin.}^2\,\phi)}$ by Δ, and alſo by $\Delta\ (\phi)$ and by $\Delta : (c,\ \phi)$, according as it is conſidered as a function of ϕ only, or as a function of c and ϕ.

Let us in the firſt place conſider the caſe in which Q is an integer function, and let $\text{Q} = \text{A}+\text{B}\,\text{ſin.}^2\,\phi+\text{C}\,\text{ſin.}^4\,\phi+\&\text{c}.$

If, for the ſake of abridging, we repreſent the integral $\int \frac{\text{ſin.}^{2k}\varphi d\varphi}{\Delta}$ by Z^{2k}, we have $\int \frac{Q dx}{\Delta} = AZ^0 + BZ^2 + CZ^4 + \&c.$

But, by taking the differential of the quantity $\Delta\, \text{coſ.}\, \varphi\, \text{ſin.}^{2k-3}\varphi$, we readily obtain the formula $\Delta\, \text{coſ.}\, \varphi\, \text{ſin.}^{2k-3}\varphi = (2k-3) \times Z^{2k-4} - (1 + c^2)(2k-2) Z^{2k-2} + c^2 (2k-1) Z^{2k}$; Hence it follows that Z^4, Z^6, &c. may be expreſſed by means of Z^0 and Z^2, and therefore, that in the caſe when Q is an integer function, the integral $\int \frac{Q d\varphi}{\Delta}$ is equal to an algebraic quantity, *plus* an integral of the form $\int (A + B\, \text{ſin.}^2 \varphi) \frac{d\varphi}{\Delta}$.

9. Now let Q be any rational function of $\text{ſin.}^2 \varphi$. After having ſeparated the integer part, and reduced it, in the manner which has been explained, the developement of the fractional part may always be ſo ordered, that every partial fraction ſhall be of the form $\frac{N}{(1 + n\, \text{ſin.}^2\varphi)^k}$, n and N being conſtant coefficients, real or imaginary. We muſt now endeavour to reduce the formula $\int \frac{d\varphi}{(1 + n\, \text{ſin.}^2\varphi)^k \Delta}$, and ſuch like, to the moſt ſimple formulas of the ſame kind. But if we make for a little, ſine $\varphi = x$, the formula becomes

$$\int \frac{dx}{(1 + nx^2)^k \sqrt{(1 - (1 + c^2) x^2 + c^2 x^4)}}.$$

Let us conſider, for the ſake of greater generality, the formula

$$\Pi^k = \int \frac{dx}{(1 + nx^2)^k R},$$

in which R repreſents the radical $\sqrt{(\alpha + \beta x^2 + \gamma x^4)}$. By differentiation, we find

$$\frac{xR}{(1 + nx^2)^{k-1}} = (2k-2)\left(\alpha - \frac{\beta}{n} + \frac{\gamma}{n^2}\right)\Pi^k - (2k-3)\left(\alpha - \frac{2\beta}{n} + \frac{3\gamma}{n^2}\right) \times \Pi^{k-1} + (2k-4)\left(-\frac{2\beta}{n} + \frac{3\gamma}{n^2}\right)\Pi^{k-2} - (2k-5)\frac{\gamma}{n^2}\Pi^{k-3}.$$

Hence

Hence it follows that the term Π^k, and all ſuch, in which k is greater than unity, may be expreſſed in part algebraically, and in part by the quantities Π^1, Π^0, Π^{-1}, which are the moſt ſimple of their kind. It may even be obſerved that if $1 + nx^2$ be a diviſor of the quantity under the radical, in which caſe we have $\alpha - \frac{\beta}{n} + \frac{\gamma}{n^2} = 0$, then the preceding formula has a term leſs, and thus the quantity Π^k depends only on theſe two Π^0 and Π^{-1}.

Therefore the determination of Π^k depends in general on the formula

$$\int\left(\frac{A}{1 + nx^2} + B + Cx^2\right)\frac{dx}{R},$$

and in particular, when $1 + nx^2$ is a diviſor of R^2, the denominator may be made to diſappear, and the determination of Π^k will then depend only on the formula $\int(B + Cx^2)\,\frac{dx}{R}$.

10. The application of this reſult to the integral $\int\frac{d\varphi}{(1 + n\ \text{ſin.}^2\ \varphi)^k\,\Delta}$ ſhews, that in general this integral depends upon the following

$$\int\left(\frac{A}{1 + n\ \text{ſin.}^2\varphi} + B + C\ \text{ſin.}^2\ \varphi\right)\frac{d\varphi}{\Delta}.$$

And in the particular caſe, in which $1 + n\ \text{ſin.}^2\ \varphi$ is a factor of $\text{coſ.}^2\varphi\,\Delta^2$, the integral may be freed from the denominator, and will then depend entirely on the integer formula

$$\int(B + C\ \text{ſin.}^2\ \varphi)\,\frac{d\varphi}{\Delta}.$$

This particular caſe occurs when $n = -1$, alſo when $n = -c^2$; then the formulas are $\int\frac{d\varphi}{\text{coſ.}^{2k}\varphi\,\Delta}$ and $\int\frac{d\varphi}{\Delta^{2k+1}}$, and indeed it is eaſy to reduce both to an algebraic part together with an integral of the form $\int(A + B\ \text{ſin.}^2\ \varphi)\,\frac{d\varphi}{\Delta}$. We may add to theſe two caſes that of $\int\frac{d\varphi}{\text{ſin.}^{2k}\varphi\,\Delta}$ which is ſuſceptible of a ſimilar reduction, and which may be effected by the formula of article 8.

It follows from this analyſis that the general formula $\int \frac{Q d\phi}{\Delta}$ when reduced to a convenient form contains; 1°, an algebraic part; 2°, an integral of the form $\int (A + B \text{ ſin.}^2 \phi) \frac{d\phi}{\Delta}$; 3°, one or more parts of the form $\int \frac{N}{1 + n \text{ ſin.}^2 \phi} \cdot \frac{d\phi}{\Delta}$, in each of which the coefficients N and n have any values whatever, real or imaginary.

Definition of Elliptic Tranſcendentals and their diviſion into three Species.

11. Since the tranſcendentals we have conſidered are always reducible to theſe two forms $\int (A + B \text{ ſin.}^2 \phi) \frac{d\phi}{\Delta}$, $\int \frac{d\phi}{(1 + n \text{ſin.}^2 \phi)\Delta}$, it is evident that they are comprehended in the general formula

$$H = \int \frac{A + B \text{ ſin.}^2 \phi}{1 + n \text{ ſin.}^2 \phi} \cdot \frac{d\phi}{\Delta}.$$

We ſhall in future call the integrals contained in this formula *elliptic functions* or *elliptic tranſcendentals.* We ſhall ſuppoſe the tranſcendental H to vaniſh, or to begin, when $\phi = 0$; and its extent to be determined by the variable arc ϕ, which we ſhall call its ***amplitude;*** the conſtant quantity c, which is always leſs than unity, ſhall be called the *modulus;* and, laſtly, $\surd(1 - c^2)$ which we ſhall always denote by b, ſhall be called the ***complement of the modulus.***

The quantity H, when ϕ is conſidered as the only variable quantity in it, may be denoted by H (ϕ); if however the modulus and amplitude be both conſidered as variable, it may be denoted by H (c, ϕ); and ſo on, if its coefficients be ſuppoſed to change their values.

12. If every value of H, from $\phi = 0$ to $\phi = 90° = \frac{\pi}{2}$, were known, we could eaſily determine the value of the tranſcendental for any other amplitude: for let $\phi = i\pi \pm \alpha$, i being a whole number, and α an arch leſs than $\frac{\pi}{2}$, then,

$$H(\phi) = 2i H\left(\frac{\pi}{2}\right) \pm H(\alpha).$$

In this reſpect the function H, agrees with elliptic arcs, and in general with the arcs of all oval curves compoſed of four equal and ſimilar parts, for any arc of theſe, however great, and containing even ſeveral circumferences, may always be expreſſed without difficulty, by the fourth of the curve and a certain portion of that fourth. The determinate function $H\left(\frac{\pi}{2}\right)$ may therefore be conſidered as the unit of the functions H, and we ſhall denote it by H^1.

It does not appear that the function H, taken in all its generality, can be reduced to elliptic arcs; this can only happen when $n = 0$, or when by ſome ſubſtitution the denominator $1 + n\,\text{ſin.}^2\,\varphi$ is made to diſappear, a reduction which we do not ſuppoſe to be always poſſible.

Thus the term *elliptic function* is in ſome reſpects improper; we retain it however becauſe of the great analogy which is found to ſubſiſt between the properties of this function, and thoſe of elliptic arcs.

13. When $n = 0$, or when by a ſuitable ſubſtitution the denominator $1 + n\,\text{ſin}^2\,\varphi$ vaniſhes, then the function H is rendered conſiderably more ſimple. In this caſe it will be proper to denote it by another character G, thus we ſhall henceforth make

$$G = \int \frac{(A + B\,\text{ſin.}^2\,\varphi)\,d\varphi}{\sqrt{(1 - c^2\,\text{ſin.}^2\,\varphi)}}.$$

The function G, as we ſhall hereafter demonſtrate, may always be readily expreſſed by two elliptic arcs, and with reſpect to elliptic arcs themſelves, they are repreſented by the formula

$$E = \int d\varphi\,\sqrt{(1 - c^2\,\text{ſin.}^2\,\varphi)},$$

concerning which, ſee *Memoirs of the Academy for* 1786.

In ſhort, if regard be had to analytical ſimplicity, the function $\int \frac{d\varphi}{\sqrt{(1 - c^2\,\text{ſin.}^2\,\varphi)}}$ is very remarkable, and it is proper to deſign it by a particular ſymbol, we ſhall therefore make

$$F = \int \frac{d\varphi}{\sqrt{(1 - c^2\,\text{ſin.}^2\,\varphi)}};$$

this function F may much rather be expreſſed by two elliptic arcs, but it may alſo be expreſſed by the arc E, and its partial difference relative to the modulus c; for it is evident that

$$F = E - c\,\frac{dE}{dc}.$$

We even believe that this expreſſion would be the moſt convenient for the numerical calculation of integrals, if there were tables of elliptic arcs, which ſhould alſo exhibit the values of the coefficient $\frac{dE}{dc}$ at the ſide of each arc.

However this may be, the quantity F may certainly be expreſſed by E, but not E by F, and much leſs G by F, and this appears to me to be a ſure proof that the function F, conſidered analytically, is more ſimple than elliptic arcs. If we judge of hyperbolic arcs in the ſame way it will appear that they are abſolutely of the ſame nature as elliptic arcs, for like as an hyperbolic arc can be expreſſed by two elliptic arcs, ſo, on the contrary, an elliptic arc can be expreſſed by two hyperbolic arcs, this will be evident from the values we ſhall afterwards give for theſe arcs, and which may alſo be found in the Memoirs for 1786 already quoted.

14. There are therefore three diſtinct ſpecies of elliptic functions or tranſcendentals. The firſt, and moſt ſimple, is repreſented by the formula

$$F = \int \frac{d\phi}{\Delta}.$$

The ſecond is repreſented by the formula

$$G = \int (A + B\ \text{ſin.}^2\,\phi)\,\frac{d\phi}{\Delta}.$$

This contains, as a particular caſe, the elliptic arc $E = \int d\phi\,\Delta$, and the hyperbolic arc $\Upsilon = \Delta\ \text{tan.}\ \phi - \int (1 - \text{ſin.}^2\,\phi)\,\frac{c^2\,d\phi}{\Delta}$.

Laſtly, the third, and moſt complex, is expreſſed by the formula

$$H = \int \frac{A + B\ \text{ſin.}^2\,\phi}{1 + n\ \text{ſin.}^2\,\phi}\cdot\frac{d\phi}{\Delta}.$$

In this laſt the coefficients A, B, n may be imaginary, but then the formula H will be accompanied by a ſimilar formula, which will differ from it only by the ſign of $\sqrt{-1}$, and the ſum of the two will always give a real reſult.

We might alſo, for the ſake of avoiding imaginary quantities, conſider the formula

$$K = \int \frac{A + B\ \text{ſin.}^2\,\phi + C\ \text{ſin.}^4\,\phi}{1 + 2n\ \text{coſ.}\ \alpha\ \text{ſin.}^2\,\phi + n^2\ \text{ſin.}^4\,\phi}\cdot\frac{d\phi}{\Delta};$$

as a diſtinct ſpecies of elliptic functions, but we do not think it neceſſary to admit this fourth ſpecies of tranſcendentals; and we ſhall confine our attention to the three others, becauſe the calculations by which the value of H is determined are the ſame whether n be real, or imaginary. It is however indiſpenſably neceſſary that the modulus c, and the amplitude φ be both real, and that at the ſame time c be leſs than unity.

Compariſon of elliptic functions of the firſt kind.

15. Every geometer knows the complete algebraic integral which Euler has given of the equation

$$\frac{dx}{\sqrt{(\alpha+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4)}}+\frac{dy}{\sqrt{(\alpha+\beta y+\gamma y^2+\delta y^3+\varepsilon y^4)}}=0.$$

This integral will be of great uſe in our inveſtigations. And in the firſt place, in conſequence of the foregoing transformations, this equation may, without loſing its generality, be put under the form

$$\frac{d\varphi}{\sqrt{(1-c^2\,\text{ſin.}^2\,\varphi)}}+\frac{d\psi}{\sqrt{(1-c^2\,\text{ſin.}^2\,\psi)}}=0;$$

its integral then is $F(\varphi)+F(\psi)=$ *conſt.* But the ſame integral found by Euler's method, and reduced to the moſt ſimple form, is

$$\text{coſ.}\,\varphi\,\text{coſ.}\,\psi-\text{ſin.}\,\varphi\,\text{ſin.}\,\psi\,\sqrt{(1-c^2\,\text{ſin.}^2\,\mu)}=\text{coſ.}\,\mu\,..(\alpha').$$

Here μ is the arbitrary conſtant quantity, and as it appears that putting $\varphi=0$, then $\psi=\mu$, therefore the firſt integral becomes

$$F(\varphi)+F(\psi)=F(\mu).$$

Therefore, as often as the angles φ, ψ, μ have among themſelves the relation expreſſed by the equation (α'), the function $F(\mu)$ ſhall be equal to the ſum of the two $F(\varphi)$, $F(\psi)$. We ſhall here leave the conſideration of the differential equation, that we may examine the conſequences which follow from this reſult, with reſpect to elliptic functions of the firſt ſpecies.

16 Having given any two elliptic functions $F(\varphi)$, $F(\psi)$, a third function $F(\mu)$ may be found equal to their ſum; for this purpoſe it will be neceſſary to deduce the value of μ from the equation (α'). But, by denoting always $\sqrt{(1-c^2\,\text{ſin.}^2\,\varphi)}$ by Δ or $\Delta(\varphi)$, and the like radical of ψ by $\Delta(\psi)$, we find

$$\text{coſ.}\,\mu=\frac{\text{coſ.}\,\varphi\,\text{coſ.}\,\psi-\Delta(\varphi)\,\Delta(\psi)\,\text{ſin.}^2\,\varphi\,\text{ſin.}\,\psi}{1-c^2\,\text{ſin.}^2\,\varphi\,\text{ſin.}^2\,\psi},$$

$$\text{ſin.}\,\mu=\frac{\Delta(\psi)\,\text{ſin.}\,\varphi\,\text{coſ.}\,\psi+\Delta(\varphi)\,\text{ſin.}\,\psi\,\text{coſ.}\,\varphi}{1-c^2\,\text{ſin.}^2\,\varphi\,\text{ſin.}^2\,\psi},$$

$$\Delta(\mu)=\frac{\Delta(\varphi)\,\Delta(\psi)-c^2\,\text{ſin.}\,\varphi\,\text{ſin.}\,\psi\,\text{coſ.}\,\varphi\,\text{coſ.}\,\psi}{1-c^2\,\text{ſin.}^2\,\varphi\,\text{ſin.}^2\,\psi},$$

$$\text{tan.}\,\mu=\frac{\Delta(\psi)\,\text{tan.}\,\varphi+\Delta(\varphi)\,\text{tan.}\,\psi}{1-\Delta(\varphi)\,\Delta(\psi)\,\text{tan.}\,\varphi\,\text{tan.}\,\psi}.$$

It may be remarked with regard to this last formula, that if two angles φ' and ψ' be taken, such, that

$$\tan. \varphi' = \Delta\,(\psi) \tan. \varphi,\ \tan. \psi' = \Delta\,(\varphi) \tan. \psi,$$

then it follows, that

$$\mu = \varphi' + \psi',$$

and thus we have an easy method of calculating the angle μ, by the tables of sines, when φ and ψ are known. It is necessary however, in the first place, to calculate $\Delta(\varphi)$ and $\Delta(\psi)$; for this purpose, take the angles α and β, such that $\sin. \alpha = c \sin. \varphi$, and $\sin. \beta = c \sin. \psi$, then will $\Delta\,(\varphi) = \cos. \alpha$, and $\Delta(\psi) = \cos. \beta$. By these formulas, we can find the amplitude μ of the elliptic function, which is the sum of two functions whose amplitudes are φ and ψ. We may thence also find the amplitude of a function which is the difference of two other functions; let

$$\mathrm{F}\,(\varphi) - \mathrm{F}\,(\psi) = \mathrm{F}\,(\mu);$$

and the amplitude μ of the function, which is the difference, is determined by any one of the following equations:

$$\cos. \mu = \frac{\cos. \varphi \cos. \psi + \Delta\,(\varphi)\ \Delta\,(\psi) \sin. \varphi \sin. \psi}{1 - c^2 \sin.^2 \varphi \sin.^2 \psi},$$

$$\sin. \mu = \frac{\Delta\,(\psi) \sin. \varphi \cos. \psi - \Delta\,(\varphi) \sin. \psi \cos. \varphi}{1 - c^2 \sin.^2 \varphi \sin.^2 \psi},$$

$$\Delta\,(\mu) = \frac{\Delta\,(\varphi)\,\Delta\,(\psi) + c^2 \sin. \varphi \sin. \psi \cos. \varphi \cos. \psi}{1 - c^2 \sin.^2 \varphi \sin.^2 \psi},$$

$$\tan. \mu = \frac{\Delta\,(\psi) \tan. \varphi - \Delta\,(\varphi) \tan. \psi}{1 + \Delta\,(\varphi)\,\Delta\,(\psi) \tan. \varphi \tan. \psi},$$

in this last, if we make $\tan. \varphi' = \Delta\,(\psi) \tan. \varphi$, and $\tan. \psi' = \Delta\,(\varphi) \tan. \psi$, we shall have $\mu = \varphi' - \psi'$.

These formulas for the difference follow from those for the sum, by simply changing the sign of ψ in the latter, and keeping $\Delta(\psi)$ positive. It is besides needless to point out the analogy which subsists between the values of $\cos. \mu$, $\sin. \mu$, and those of $\cos. (\varphi \pm \psi)$, $\sin. (\varphi \pm \psi)$; they coincide entirely if c is supposed $= 0$.

17. Since we know algebraically, the amplitude of the function equal to the sum, or to the difference of two given functions, it is evident that we can find, algebraically, a multiple elliptic function of a given function, and that in general we can resolve the same problems relating to the multiplication and division of elliptic functions of the first kind, as can be resolved relating to the multiplication and division of arcs of a circle. Let φn denote the amplitude of the function which contains n times the function of which the amplitude is φ, so that we have $\mathrm{F}\,(\varphi n) = n\,\mathrm{F}(\varphi)$; it may be required to determine the general expression of

sin. or cos. φn, by means of sin. and cos. φ. The following are some researches relative to this subject.

The extreme cases present no difficulty. If $c = 0$, then $\varphi n = n\varphi$, and we have the known relation, $\cos. \varphi n + \sin. \varphi n \times \sqrt{-1} = (\cos. \varphi + \sin. \varphi \sqrt{-1})^n$. If $c = 1$, then the function $F(\varphi)$ becomes $\int \frac{d\varphi}{\cos. \varphi}$, or $\frac{1}{2} \log. \frac{1 + \sin. \varphi}{1 - \sin. \varphi}$, so that, for the multiplication of functions, we have this very simple relation, $\frac{1 + \sin. \varphi n}{1 - \sin. \varphi n} = \left(\frac{1 + \sin. \varphi}{1 - \sin. \varphi}\right)^n$. The difficulty is to find the general expression of sin. φn when c has any value between 0 and 1.

Let us in the first place consider the most simple case, which is that of duplication, then we have

$$\cos. \varphi 2 = \frac{1 - 2 \sin.^2 \varphi + c^2 \sin.^4 \varphi}{1 - c^2 \sin.^4 \varphi},$$

$$\sin. \varphi 2 = \frac{2\Delta(\varphi) \sin. \varphi \cos. \varphi}{1 - c^2 \sin.^4 \varphi},$$

$$\tan. \tfrac{1}{2}\varphi 2 = \Delta(\varphi) \tan. \varphi.$$

The last of these formulas appears convenient for trigonometrical calculation; for if we take the angles α, $\alpha 2$, $\alpha 4$, such that $\sin. \alpha = c \sin. \varphi$, $\sin. \alpha 2 = c \sin. \varphi 2$, $\sin. \alpha 4 = c \sin. \varphi 4$, &c. it follows that

$$\tan. \tfrac{1}{2}\varphi 2 = \cos. \alpha \tan. \varphi, \quad \tan. \tfrac{1}{2}\varphi 4 = \cos. \alpha 2 \tan. \varphi 2, \text{ \&c.}$$

so that the function F shall be multiplied by 2, 4, 8, 16, &c.

If it be required to divide by the same numbers; let $\varphi\frac{1}{2}$ be put, by analogy, to denote the amplitude of the function which is the half of F, and the inverse of the preceding formulas give

$$\sin. \varphi \tfrac{1}{2} = \frac{\sin. \frac{1}{2}\varphi}{\sqrt{\left(\frac{1 + \Delta}{2}\right)}}.$$

Now let $\sin. \alpha = c \sin. \varphi$, then $\Delta = \cos. \alpha$, and we have this very simple expression,

$$\sin. \varphi \tfrac{1}{2} = \frac{\sin. \frac{1}{2}\varphi}{\cos. \frac{1}{2}\alpha};$$

If in like manner, we make $\sin. \alpha\frac{1}{2} = c \sin. \varphi\frac{1}{2}$, we shall have

$$\sin. \varphi \tfrac{1}{4} = \frac{\sin. \frac{1}{2}\varphi\frac{1}{2}}{\cos. \frac{1}{2}\alpha\frac{1}{2}},$$

and so on.

18. Let us now proceed to the multiplication and division by any number. For this purpose, let us consider the three conse-

cutive amplitudes $\varphi\overline{n-1}$, φn, $\varphi\overline{n+1}$, which agreeably to the preceding notation, answer to the functions $(n-1)$ F, nF, $(n+1)$ F. The formulas for the sum and the difference of two functions apply to the equations $F(\varphi\overline{n+1}) = F(\varphi n) + F(\varphi)$, $F(\varphi\overline{n-1}) = F(\varphi n) - F(\varphi)$, and hence it follows, that

$$\sin.\varphi\,\overline{n+1} + \sin.\varphi\,\overline{n-1} = \frac{2\Delta\cos.\varphi\sin.\varphi n}{1-c^2\sin.^2\varphi\sin.^2\varphi n};$$

$$\cos.\varphi\,\overline{n+1} + \cos.\varphi\,\overline{n-1} = \frac{2\cos.\varphi\cos.\varphi n}{1-c^2\sin.^2\varphi\sin.^2\varphi n}.$$

These formulas in which Δ and φ remain always the same, while n varies, appear to be the best possible for giving the successive values of sin. $\varphi 2$, sin. $\varphi 3$, cos. $\varphi 2$, cos. $\varphi 3$, &c. Thus, putting (with a view to abridge), $2\Delta\cos.\varphi = p$, $1 - c^2\sin.^4\varphi = q$, $c^2\sin.^4\varphi = 1 - q = r$, we have

$$\sin.\varphi 2 = \frac{p}{q}\sin.\varphi,$$

$$\sin.\varphi 3 = \frac{p^2-q^2}{q^2-rp^2}\sin.\varphi,$$

$$\sin.\varphi 4 = \frac{(1+r)p^2-2q^2}{q^4-rp^4}\,pq\sin.\varphi,$$

$$\sin.\varphi 5 = \frac{q^6-3p^2q^4+(1+2r)p^4q^2-r^2p^6}{q^6-3rp^2q^4+r(r+2)p^4q^2-rp^5}\sin.\varphi,$$

&c.

But these expressions, when completely developed, become extremely complicated, and their law is very difficult to discover.

When it is proposed to calculate trigonometrically φn by means of φ, it may be easily done as follows. The two foregoing formulas give by division

$$\frac{\sin.\varphi\,\overline{n+1} + \sin.\varphi\,\overline{n-1}}{\cos.\varphi\,\overline{n+1} + \cos.\varphi\,\overline{n-1}} = \Delta\tan.\varphi n,$$

which is reducible to this very simple formula,

$$\tan.(\tfrac{1}{2}\varphi\,\overline{n+1} + \tfrac{1}{2}\varphi\,\overline{n-1}) = \Delta\tan.\varphi n;$$

and hence may be deduced successively

$$\tan.\tfrac{1}{2}\varphi 2 = \Delta\tan.\varphi,$$
$$\tan.(\tfrac{1}{2}\varphi 3 + \tfrac{1}{2}\varphi) = \Delta\tan.\varphi 2,$$
$$\tan.(\tfrac{1}{2}\varphi 4 + \tfrac{1}{2}\varphi 2) = \Delta\tan.\varphi 3,$$

&c.

Now Δ being always the same in these formulas, no simpler method of calculating the successive values of $\varphi 2$, $\varphi 3$, &c. by means of φ need be wished for. We may also proceed by greater intervals in the calculation of φm, m being as great as we please: for we have in like manner

tan.

$$\tan.\left(\tfrac{1}{2}\varphi\overline{n+i}+\tfrac{1}{2}\varphi\overline{n-i}\right)=\Delta i\tan.\varphi n,$$

Δi being the Δ which corresponds to the amplitude φi.

The division of a given elliptic function into a certain number of equal parts is an algebraic problem which may be resolved by the developement of the formulas which serve for its multiplication. We have found already the formulas for its division by 2 or any power of 2; let it now be required to divide the function F into three equal parts; let $\varphi 3$ be the given amplitude of the function F, and φ that of the function which is one-third of it. We make $\sin.\varphi 3=a$, $\sin.\varphi=x$, and the equation to be resolved for its trisection will be

$$a=\frac{3-4(1+c^2)x^2+6c^2x^4-c^4x^8}{1-6c^2x^4+4c^2(1+c^2)x^6-3c^4x^8}\cdot x.$$

This equation is of the ninth degree; it would be of the 25th degree for the quinquisection, and so on.

The equations are less complex by the half when it is proposed to divide the whole function F1, the amplitude of which is 90°. Suppose in general $\varphi n=90°$, the formula (a') of article 15, where we may make $\mu=90°$, gives immediately this relation,

$$\tan.\varphi\overline{n-i}=\tfrac{1}{b}\cot.\varphi i;$$

hence we have successively $\tan.\varphi\overline{n-1}=\frac{1}{b}\cot.\varphi$, $\tan.\varphi\overline{n-2}=\frac{1}{b}\cot.\varphi 2$, $\tan.\varphi\overline{n-3}=\frac{1}{b}\cot.\varphi 3$, &c. Again, because $\varphi n=90°$, we have also

$$\tan.\left(45°+\tfrac{1}{2}\varphi\overline{n-2}\right)=\Delta\tan.\varphi\overline{n-1}=\frac{\Delta}{b}\cot.\varphi$$

$$\tan.\left(\tfrac{1}{2}\varphi\overline{n-1}+\tfrac{1}{2}\varphi\overline{n-3}\right)=\Delta\tan.\varphi\overline{n-2}$$

&c.

from which $\varphi\overline{n-1}$, $\varphi\overline{n-2}$, &c. may be determined by formulas sufficiently simple. Developing in like manner the formulas which give $\varphi 2$, $\varphi 3$, &c. until the two series meet, we have by their union the means of obtaining an equation which will determine φ; thus when $n=3$, that equation is directly

$$\tan.\left(45°+\tfrac{1}{2}\varphi\right)=\frac{\Delta}{b}\cot.\varphi,$$

or $b\sin.\varphi=\Delta(1-\sin.\varphi)$, from which, making $\sin.\varphi=x$, we find

$$0=1-2x+2c^2x^3-c^2x^4,$$

an equation for the trisection of the function F1, and of which the degree is only 4 or $\frac{9-1}{2}$.

When $n=5$ it is necessary to eliminate $\varphi 3$ from these two equations,

$$\tan.\left(45°+\tfrac{1}{2}\varphi 3\right)=\frac{\Delta}{b}\cot.\varphi,$$

tan.

$$\tan.\ (\tfrac{1}{2}\varphi 3 + \tfrac{1}{2}\varphi) = \Delta\ \tan.\ \varphi 2,$$

and then to substitute for tan. $\varphi 2$ its value

$\frac{2\Delta \sin.\ \varphi \cos.\ \varphi}{1 - 2\sin.^2 \varphi + c^2 \sin.^4 \varphi}$. We thus obtain, making $\sin.\ \varphi = x$,

$$\frac{(1+x)\Delta}{bx} = \frac{1 + 2x - 2c^2x^3 - c^2x^4}{1 - 2x + 2c^2x^3 - c^2x^4},$$

an equation for the quinquisection of the function F1, and which when completely developed rises to the 12th or $\frac{25-1}{2}$th degree.

Such are the formulas by which we may find the relation between φn and φ in order that $F(\varphi n) = nF(\varphi)$, n being a whole number. If it were required to find the relation between φ and ψ, so that $F(\psi) = \frac{m}{n}F(\varphi)$, m and n being integers, an auxiliary angle ω might be taken, such, that $nF(\psi) = F(\omega)$, and $mF(\varphi) = F(\omega)$. The first condition would give an algebraical equation between the sines and cosines of the angles ψ and ω; and the second an equation between the angles φ and ω, whence, elimininating ω, the required relation between φ and ψ would be found.

19. Hence it follows that we may always find the complete algebraic integral of the equation

$$\frac{md\varphi}{\sqrt{(1 - c^2\sin.^2\varphi)}} + \frac{nd\psi}{\sqrt{(1 - c^2\sin.^2\psi)}} = 0,$$

m and n being whole numbers. For the integral is in the first place $mF(\varphi) + nF(\psi) = const.$ and if we suppose that when $\varphi = 0$, $\psi = \mu$, the constant quantity will be $nF(\mu)$, and thus we have

$$mF(\varphi) + nF(\psi) = nF(\mu).$$

But an algebraic equation can be found which shall represent this transcendental equation, for if we make $F(\omega) = F(\mu) - F(\psi)$, which gives $mF(\varphi) = nF(\omega)$, each of these equations may be expressed in algebraic terms, and thus, by eliminating ω, we shall have the complete algebraic integral of the proposed equation.

In general, in the following equation

$$0 = \frac{md\varphi}{\sqrt{(1 - c^2\sin.^2\varphi)}} + \frac{nd\psi}{\sqrt{(1 - c^2\sin.^2\psi)}} + \frac{pd\omega}{\sqrt{(1 - c^2\sin.^2\omega)}} + \&c.$$

consisting of any finite number of terms, and in which m, n, p, &c. are whole numbers positive or negative, the complete integral will be $F(\mu) = mF(\varphi) + nF(\psi) + pF(\omega) + \&c.$ μ being the arbitrary constant quantity: but this transcendental equation may always be changed into an algebraic equation; for making

F

$F(\varphi') = mF(\varphi)$, $F(\psi') = nF(\psi)$, $F(\omega') = pF(\omega)$, &c. so that $F(\mu) = F(\varphi') + F(\psi') + F(\omega') +$ &c. each of these equations may be changed into an algebraic equation, and if we eliminate φ', ψ', ω', &c. the resulting equation will contain the arbitrary quantity μ, and will be the complete algebraic integral of the proposed equation.

If we denote the radical $\sqrt{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4)}$ by $R(x)$, and a like radical of y by $R(y)$, &c. and suppose m, n, p, &c. as before whole numbers, positive or negative, it is evident that the equation

$$0 = \frac{m dx}{R(x)} + \frac{n dy}{R(y)} + \frac{p dz}{R(z)} + \text{\&c.}$$

may always be reduced to the preceding form, and consequently will always have a complete algebraic integral.

We may suppose that z and the following variable quantities are given algebraic functions of x and y; then the preceding equation will contain only two variable quantities, and notwithstanding the infinity of forms of which it is susceptible, it will always admit of a complete algebraic integral. It appears to me that this manner of generalizing the result of Euler relating to the equation $\frac{dx}{R(x)} + \frac{dy}{R(y)} = 0$, has not yet been attended to. A celebrated geometer has proposed (*Mem. de Turin, Tome IV.*) to find cases in which the equation $\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$, is integrable, X being supposed any polynomial formed from x, and Y a like polynomial from y; but it does not appear, that his researches have gone beyond the equation of Euler; for the equation which he gives, (page 119.) as more general, is immediately reduced to it by making $v = ky$, and giving a suitable value to k. Thus it is very doubtful, whether the equation of Euler can be made more general with two terms only, but with more than two it appears that the equation may be greatly extended.

20. Since the functions F admit of being multiplied or divided by any number, we may, by means of this property, find their values by approximation. In the first place, we may suppose that φ does not exceed 90°, for we have shewn, (art. 12,) that every other case is reducible to this one. This being assumed, we may, art. 17, determine $\varphi\frac{1}{2}$, so that $F(\varphi\frac{1}{2}) = \frac{1}{2}F\varphi$, and the extent of the integral $\int \frac{d\varphi\frac{1}{2}}{\Delta(\varphi\frac{1}{2})}$ will be nearly half that of $\int \frac{d\varphi}{\Delta(\varphi)}$, if $c \sin. \varphi$ is not very near to unity. By a second bisec-

tion; we may take $F\ (\varphi\frac{1}{4}) = \frac{1}{4}F\ (\varphi)$, and the integral $\int \frac{d\varphi\frac{1}{4}}{\Delta\ (\varphi\frac{1}{4})}$ will have only one fourth, nearly, of that extent, and so on. But when the amplitude φ is greatly diminished, $F\ (\varphi)$ is reduced nearly to the arc φ. Therefore, whatever be the first value of φ, the corresponding function $F\ (\varphi)$ is equal to the last term of the series $2\varphi\frac{1}{2}$, $4\varphi\frac{1}{4}$, $8\varphi\frac{1}{8}$, &c. and in most cases we may find this limit by a very short calculation.

EXAMPLE.

Let it be proposed to find the value of F when $c = \sqrt{(\frac{2+\sqrt{3}}{4})} = \sin.\ 75^\circ$, and $\tan.\ \varphi = \sqrt{(\frac{2}{\sqrt{3}})}$. By employing the formulas of art 17, and working by Gardiner's Tables, we shall find

$\varphi = 47^\circ\ 3'\ 30''.91$
$\varphi\frac{1}{2} = 25\ 36\ 5\ .64$
$\varphi\frac{1}{4} = 13\ 6\ 30\ .985$
$\varphi\frac{1}{8} = 6\ 35\ 40\ .741$
$\varphi\frac{1}{16} = 3\ 18\ 8\ .748$
&c.

$\alpha = 45^\circ\ 0'\ 0'.00$
$\alpha\frac{1}{2} = 24\ 40\ 10.94$
$\alpha\frac{1}{4} = 12\ 39\ 15.83$
$\alpha\frac{1}{8} = 6\ 22\ 8.4$
&c.

The two last values give

$$8\varphi\tfrac{1}{8} = 52^\circ\ 45'\ 25''.93$$
$$16\varphi\tfrac{1}{16} = 52\ 50\ 19\ .97.$$

Their difference is $4'\ 54''.04$; and, as by the nature of these approximations, a result ought to fall short of the limit by about one-fourth of the distance of the preceding one from it, we add to the last result the third of the difference $4'\ 54''.04$, which is $1'\ 38''.01$, and thus get F very nearly equal to the arc $52^\circ\ 51'\ 57''.98$, which, in parts of the radius is $\frac{\pi}{2} \times 0.5874012 = 0.9226877$.

This method may in some cases, be very tedious: we shall in the sequel give another much more expeditious.

A remark upon the motion of a simple pendulum.

21. We have shewn in the *Memoirs of the Academy* for 1786, page 637, that the motion of a simple pendulum admits of two cases; in the one it oscillates in the vertical plane, and in the

other, it turns always in the same direction. But as the two cases lead to the same formula, we shall only consider the case of oscillations. Therefore, the force of gravity being unity, let l be the length of the pendulum, h the height due to the velocity at the lowest point, φ the angle which the pendulum makes with the vertical at the end of the time t. Also, let $\frac{h}{2l} = c^2$, and $\sin. \frac{1}{2}\varphi = c \sin. \psi$, then $dt = \sqrt{l} . \frac{d\psi}{\sqrt{(1 - c^2 \sin.^2 \psi)}}$, and consequently $t = \sqrt{l}.\mathrm{F}\,(\psi)$.

Thus the time taken to describe any arc is an elliptic function of the first kind, and it has all the properties of these functions. Hence it follows, that the arc described in any time t being given, we can find algebraically another arc described in a time which is a multiple of t, or in general, which is commensurable with t. We can also find an arc, such, that the time of describing it shall be equal to the sum or to the difference of the times of describing several other given arcs, and this whether the arcs be adjoining to the vertical or not. These properties are new and remarkable, and it does not appear, that any one has observed that we can compare algebraically the times of portions of oscillation like as we compare circular arcs. Thus to give a very simple example, let it be proposed to divide into two equal parts the time of a semi-oscillation. Let φ be the arc reckoned from the vertical to the point which answers to the middle of the semi-oscillation; then I say that the chord of φ shall be to the chord of $\frac{1}{2}a$ as $\sqrt{2}$ is to 1. For since $\mathrm{F}\,(\psi) = \frac{1}{2}\mathrm{F}\,(90°)$, the formula of art. 17 gives $\sin.^2 \psi = \frac{1}{1 + \sqrt{(1 - c^2)}}$, and therefore $\sin.^2 \frac{1}{2}\varphi = \frac{c^2}{1 + \sqrt{(1 - c^2)}}$; but it is evident that $c = \sin. \frac{1}{2}a$; therefore $\sin.^2 \frac{1}{2}\varphi = 1 - \cos. \frac{1}{2}a = 2 \sin.^2 \frac{1}{4}a$, and hence $2 \sin. \frac{1}{2}\varphi : 2 \sin. \frac{1}{4}a :: \sqrt{2} : 1$.

Comparison of Elliptic Functions of the second kind.

22. Let us consider two elliptic functions of the second species, which only differ in respect of their amplitudes φ and ψ, so that we have

$$\mathrm{G}\,(\varphi) = \int \frac{(\mathrm{A} + \mathrm{B} \sin.^2 \varphi) d\varphi}{\sqrt{(1 - c^2 \sin.^2 \varphi)}},\ \mathrm{G}\,(\psi) = \int \frac{(\mathrm{A} + \mathrm{B} \sin.^2 \psi)\, d\psi}{\sqrt{(1 - c^2 \sin.^2 \psi)}}.$$

Let us suppose that the angles φ and ψ have to each other the relation expressed by the equation $\mathrm{F}\,(\varphi) + \mathrm{F}\,(\psi) = \mathrm{F}\,(\mu)$, μ

being a constant quantity, which gives

$$\frac{d\varphi}{\sqrt{(1-c^2\sin.^2\varphi)}} + \frac{d\psi}{\sqrt{(1-c^2\sin.^2\psi)}} = 0;$$

hence there results $G(\varphi) + G(\psi) = \int \frac{B(\sin.^2\varphi - \sin.^2\psi)\,d\varphi}{\sqrt{(1-c^2\sin.^2\varphi)}}$ $+ \mathit{const.}$ But when $\varphi = 0$, then $\psi = \mu$; thus, supposing the integral taken from $\varphi = 0$, we have

$$G(\varphi) + G(\psi) = G(\mu) + \int B(\sin.^2\varphi - \sin.^2\psi)\frac{d\varphi}{\Delta(\varphi)}.$$

But the equation $F(\varphi) + F(\psi) = F(\mu)$, is in effect, the same as equation (α') of article 15, and from which we find, after some reductions,

$$d\varphi\Delta(\varphi) + d\psi\Delta(\psi) = c^2\sin.\mu d(\sin.\varphi\sin.\psi).$$

Besides, we have $\frac{d\varphi}{\Delta(\varphi)} + \frac{d\psi}{\Delta(\psi)} = 0$, therefore

$(\sin.^2\varphi - \sin.^2\psi)\frac{d\varphi}{\Delta} = -\sin.\mu d(\sin.\varphi\sin.\psi)$. Hence, upon the whole,

$$G(\varphi) + G(\psi) - G(\mu) = -B\sin.\mu\sin.\varphi\sin.\psi.$$

Thus it appears, that the same relation between the angles φ, ψ, μ, which gives $F(\varphi) + F(\psi) - F(\mu) = 0$, gives $G(\varphi) + G(\psi) - G(\mu)$, if not equal to 0, at least equal to a very simple algebraic quantity. This equation is the source of many consequences analogous to those which have been deduced from the function F. If we denote as before, by $\varphi 2$, $\varphi 3$, $\varphi 4$, &c. amplitudes such, that $F(\varphi 2) = 2F(\varphi)$, $F(\varphi 3) = 3F(\varphi)$, &c. we have, in consequence of this equation,

$$G(\varphi 2) - 2G(\varphi) = B\sin.\varphi\sin.\varphi\sin.\varphi 2,$$

$$G(\varphi 3) - 3G(\varphi) = B\sin.\varphi(\sin.\varphi\sin.\varphi 2 + \sin.\varphi 2\sin.\varphi 3)$$

$$G(\varphi 4) - 4G(\varphi) = B\sin.\varphi\left\{\begin{matrix}\sin.\varphi\sin.\varphi 2 + \sin.\varphi 2\sin.\varphi 3\\ + \sin.\varphi 3\sin.\varphi 4\end{matrix}\right\},$$

&c.

Therefore the same relation between φn and φ, which gives $F(\varphi n) = nF(\varphi)$, gives $G(\varphi n) - nG(\varphi)$ equal to an algebraic quantity.

23. It appears, without entering more into detail, that if m, n, p, are whole numbers, positive or negative, we may have

$$mG(\varphi) + nG(\psi) + pG(\omega) + \&c.$$

equal to an algebraic quantity. That this may be the case, it is necessary to determine the relation between the angles φ, ψ, ω, &c. which gives

$$0 = mF(\varphi) + nF(\psi) + pF(\omega) + \&c.$$

We observe that the functions $F(\varphi)$, $G(\varphi)$, have in general, the same sign as φ; when φ changes its sign, they also change their signs, retaining the same values. This being premised, we may satisfy the equation

$$0 = m\mathrm{F}(\varphi) + n\mathrm{F}(\psi) + p\mathrm{F}(\omega) + \text{\&c.}$$

in many different ways, for we have it in our power to change the sign of each coefficient, provided that the sign of the corresponding amplitude be changed at the same time. But we may limit the functions $\mathrm{F}(\varphi)$, $\mathrm{F}(\psi)$, $\mathrm{G}(\varphi)$, &c. to positive values, and in this case there is only one relation between the angles φ, ψ, ω, &c. which gives $0 = m\mathrm{F}(\varphi) + n\mathrm{F}(\psi) + p\mathrm{F}(\omega) +$ &c. then it appears that the coefficients m, n, p, &c. cannot be all positive.

Whatever has been demonstrated of the function G in general, applies to elliptic arcs in particular, and is easily extended to hyperbolic arcs. The results agree with those we have found in the volume already quoted, but the connection that subsists between these properties, is here better perceived by the relations between elliptic functions of the second species, and those of the first. We shall see immediately, that elliptic functions of the third kind, have similar properties, with this gradation, that the difference which is $= 0$ in the first kind of functions, and which is an algebraic quantity in the second kind, is a transcendental quantity in the third, but may always be determined by arcs of a circle and logarithms.

Comparison of Elliptic Functions of the third kind.

24. The general formula of these functions, is

$$\mathrm{H}(\varphi) = \int \frac{\mathrm{A} + \mathrm{B}\sin.^2\varphi}{1 + n\sin.^2\varphi} \cdot \frac{d\varphi}{\Delta(\varphi)};$$

If we consider a like function of ψ, and suppose that between φ, ψ, and the constant arc μ, the equation $\mathrm{F}(\varphi) + \mathrm{F}(\psi) - \mathrm{F}(\mu) = 0$, takes place, then because $\dfrac{d\varphi}{\Delta(\varphi)} + \dfrac{d\psi}{\Delta(\psi)} = 0$,

$$\mathrm{H}(\varphi) + \mathrm{H}(\psi) - \mathrm{H}(\mu) = \int \left\{ \frac{\mathrm{A}+\mathrm{B}\sin.^2\varphi}{1+n\sin.^2\varphi} \cdot \frac{d\varphi}{\Delta(\varphi)} + \frac{\mathrm{A}+\mathrm{B}\sin.^2\psi}{1+n\sin.^2\psi} \cdot \frac{d\psi}{\Delta(\psi)} \right\}$$

$$= \int \frac{(\mathrm{B} - \mathrm{A}n)(\sin.^2\varphi - \sin.^2\psi)}{1 + n(\sin.^2\varphi + \sin.^2\psi) + n^2\sin.^2\varphi\sin.^2\psi} \cdot \frac{d\varphi}{\Delta(\varphi)}.$$

Let $\sin.^2\varphi + \sin.^2\psi = p$, $\sin.\varphi\sin.\psi = q$; we have already found, (art. 22.), $(\sin.^2\varphi - \sin.^2\psi)\dfrac{d\varphi}{\Delta} = -\sin.\mu d(\sin.\varphi\sin.\psi)$, thus we have

$$\mathrm{H}(\varphi) + \mathrm{H}(\psi) - \mathrm{H}(\mu) = \int \frac{(\mathrm{A}n - \mathrm{B})\sin.\mu dq}{1 + np + n^2q^2};$$

But the algebraic equation between φ, ψ, μ is $\cos. \varphi \cos. \psi = \Delta (\mu) \sin. \varphi \sin. \psi + \cos. \mu$; by squaring this equation, and making the necessary substitutions, we thence deduce $p = \sin.^2 \mu - 2q \Delta (\mu) \cos. \mu + q^2 c^2 \sin.^2 \mu$, which value being put in the preceding formula, gives

$$H(\varphi)+H(\psi)-H(\mu)=\int \frac{(An - B) \sin. \mu dq}{1+n\sin.^2\mu-2nq\Delta(\mu)\cos.\mu+q^2(n^2+nc\sin^2.\mu)}$$

the integral in this expression ought to be taken from $q = 0$.

From this equation, and all those that can be formed in the same manner between three functions, we conclude that if m', n', p', &c. are whole numbers, positive or negative, we can always express

$$m'H(\varphi) + n'H(\psi) + p'H(\omega) + \&c.$$

by arcs of a circle or logarithms. To effect this, we must establish such a relation among the amplitudes φ, ψ, ω, &c. that

$$m'F(\varphi) + n'F(\psi) + p'F(\omega) + \&c. = 0.$$

This result is subject to no exception; it even takes place when the coefficients A, B, n, are imaginary. But these properties may be considered under a more general point of view.

25. Let P, or P (φ), be a rational even function of $\sin. \varphi$, and let

$$Z(\varphi) = \int \frac{P(\varphi)\, d\varphi}{\sqrt{(1 - c^2 \sin.^2 \varphi)}};$$

let Z (ψ) be a function similar to Z (φ), and let us suppose that we have always the equation $\cos. \varphi \cos. \psi = \Delta(\mu) \sin. \varphi \sin. \psi + \cos. \mu$, which, considering μ as constant, gives $\frac{d\varphi}{\Delta(\varphi)} + \frac{d\psi}{\Delta(\psi)} = 0$, we have, therefore

$$Z(\varphi) + Z(\psi) - Z(\mu) = \int \left\{ P(\varphi) - P(\psi) \right\} \frac{d\varphi}{\Delta(\varphi)}.$$

Make as before, $\sin.^2 \varphi + \sin.^2 \psi = p$, $\sin. \varphi \sin. \psi = q$, and we get

$$\sin.^2 \varphi = \tfrac{1}{2}p + \tfrac{1}{2}\sqrt{(p^2 - 4q^2)}$$
$$\sin.^2 \psi = \tfrac{1}{2}p - \tfrac{1}{2}\sqrt{(p^2 - 4q^2)}.$$

Let us now substitute the value of $\sin.^2 \varphi$ in P, the result will be of the form

$$P(\varphi) = M + N\sqrt{(p^2 - 4q^2)},$$

M and N being rational functions of p and q. It is evident, that we have, in like manner,

$$P(\psi) = M - N\sqrt{(p^2 - 4q^2)};$$

therefore $\mathrm{P}(\varphi) - \mathrm{P}(\psi) = 2\mathrm{N}\sqrt{(p^2 - 4q^2)} = 2\mathrm{N}(\sin.^2\varphi - \sin.^2\psi)$, and thus the integral $\int\left\{\mathrm{P}(\varphi) - \mathrm{P}(\psi)\right\}\frac{d\varphi}{\Delta(\varphi)} = \int 2\mathrm{N}(\sin.^2\varphi - \sin.^2\psi)\frac{d\varphi}{\Delta(\varphi)} = -2\sin.\mu\int \mathrm{N}dq$, and finally

$$\mathrm{Z}(\varphi) + \mathrm{Z}(\psi) - \mathrm{Z}(\mu) = -2\sin.\mu\int \mathrm{N}dq.$$

In this formula, N is a rational function of p and q; if we substitute in it the value of p, found in art. 24, N will be a rational function of q only, and thus the value of $\mathrm{Z}(\varphi) + \mathrm{Z}(\psi) - \mathrm{Z}(\mu)$ may always be determined by arcs of a circle and logarithms.

In general, if m', n', p', &c. denote whole numbers, positive or negative, and we establish among the angles φ, ψ, ω, the relation $0 = m'\mathrm{F}(\varphi) + n'\mathrm{F}(\psi) + p'\mathrm{F}(\omega)$, the same relation will give

$$m'\mathrm{Z}(\varphi) + n'\mathrm{Z}(\psi) + p'\mathrm{Z}(\omega) + \&c.$$

equal to a quantity determinable by arcs of a circle and logarithms. The same property also holds true even when P contains odd powers of sin. φ, for the part of Z affected by the uneven powers, is integrable by the common rules.

The very same is true of the function

$$\mathrm{Z}(x) = \int\frac{\mathrm{P}dx}{\sqrt{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4)}},$$

P being a rational function of x, and we can always find an algebraic equation between x, y, z, &c. by means of which

$$m'\mathrm{Z}(x) + n'\mathrm{Z}(y) + p'\mathrm{Z}(z) + \&c.$$

is determinable by arcs of a circle, and logarithms.

Method of Approximation applied to Elliptic Functions of the first kind.

26. The method we are to employ is the same as that we have used in the volume of 1786, already quoted, in order to reduce the rectification of a given ellipse to that of two other ellipses as little different from a circle as we please. The spirit of this method consists in reducing the integral of a differential affected with the radical $\sqrt{(1 - c^2\sin.^2\varphi)}$ to that of a like differential in which the modulus c is less than any given quantity.

In the case of the rectification of the ellipse the modulus c, represents the excentricity, and its complement $\sqrt{(1 - c^2)}$, or b is the lesser axis. We have shewn that, supposing always the semi-transverse axis $= 1$, all the ellipses which can be thus rectified by means of two of them, and which compose the same series or family, have their excentricities c, c', c'', &c. determined by the following law,

$$c' = \frac{2\sqrt{c}}{1+c},\ c'' = \frac{2\sqrt{c'}}{1+c'},\ c''' = \frac{2\sqrt{c''}}{1+c''},\ \text{\&c.}$$

In this direction, the excentricities increase continually, and approach rapidly to unity; they decrease with the same rapidity in the contrary direction. We shall denote the same series continued backwards by c, c°, $c^{\circ\circ}$, &c. so that the complete series of excentricities, which has on the one hand, 0 for a limit, and on the other unity, is

$$0 \ldots\ldots c^{\circ\circ\circ},\ c^{\circ\circ},\ c^{\circ},\ c,\ c',\ c'',\ c''', \ldots\ldots\ldots\ldots 1,$$

we have, therefore similarly,

$$c = \frac{2\sqrt{c^{\circ}}}{1+c^{\circ}},\ c^{\circ} = \frac{2\sqrt{c^{\circ\circ}}}{1+c^{\circ\circ}},\ \text{\&c.}$$

But to determine c° from c, we employ the formula $c^{\circ} = \frac{1 - \sqrt{(1 - c^2)}}{1 + \sqrt{(1 - c^2)}}$, or $c^{\circ} = \frac{1-b}{1+b}$, and thus we have

$$c^{\circ} = \frac{1 - b}{1 + b},\ c^{\circ\circ} = \frac{1 - b^{\circ}}{1 + b^{\circ}},\ c^{\circ\circ\circ} = \frac{1 - b^{\circ\circ}}{1 + b^{\circ\circ}},\ \text{\&c.}$$

It is by this law, we determine the decreasing *moduli* c°, $c^{\circ\circ}$, $c^{\circ\circ\circ}$, &c. of which we shall make frequent use in the following investigations. We may also employ the formulas

$$b^{\circ} = \frac{2\sqrt{b}}{1+b},\ b^{\circ\circ} = \frac{2\sqrt{b^{\circ}}}{1+b^{\circ}},\ \text{\&c.}$$

Lastly, we shall afterwards shew how they may be found by the tables of sines.

27. This being premised, let us consider the integral formula, $F = \int \frac{d\varphi}{\sqrt{(1 - c^2 \sin.^2 \varphi)}}$, and let there be taken a new variable arc φ°, such that

$$2\sin.\ \varphi^2 = 1 + c^{\circ} \sin.^2 \varphi^{\circ} - \Delta^{\circ} \cos.\ \varphi^{\circ};$$

we have shewn how c° is to be found from c; as to Δ°, it is put by analogy for $\sqrt{(1 - c^{\circ 2} \sin.^2 \varphi^{\circ})}$, From this supposition, we get

$$2\cos.^2 \varphi = 1 - c^{\circ} \sin.^2 \varphi^{\circ} + \Delta^{\circ} \cos.\ \varphi^{\circ},$$

$$\Delta = \frac{c^{\circ} \cos.\ \varphi^{\circ} + \Delta^{\circ}}{1 + c^{\circ}},$$

$$2 \sin.\ \varphi \cos.\ \varphi = \sin.\ \varphi^{\circ} (c^{\circ} \cos.\ \varphi^{\circ} + \Delta^{\circ});$$

Again, the differential of $2\sin.^2 \varphi$ being taken, gives

$$4\sin.\ \varphi \cos.\ \varphi d\varphi = \frac{d\varphi^{\circ} \sin.\ \varphi^{\circ}}{\Delta^{\circ}} (c^{\circ} \cos.\ \varphi^{\circ} + \Delta^{\circ})^2;$$

and dividing this equation by the last, we have

$$2d\varphi = \frac{d\varphi^{\circ}}{\Delta^{\circ}}\,(c^{\circ}\cos.\varphi^{\circ} + \Delta^{\circ}).$$

Therefore, at last we find

$$\frac{d\varphi}{\Delta} = \frac{1+c^{\circ}}{2}\cdot\frac{d\varphi^{\circ}}{\Delta^{\circ}},$$

and integrating, $\int\frac{d\varphi}{\Delta} = \frac{1+c^{\circ}}{2}\int\frac{d\varphi^{\circ}}{\Delta^{\circ}}$, or simply $F = \frac{1+c^{\circ}}{2}F^{\circ}$, where F° denotes the new elliptic function $\int\frac{d\varphi^{\circ}}{\Delta^{\circ}}$.

It is evident, that by this substitution, which gives a very simple result, the determination of the function F, is reduced to that of F°; which is more easily valued, because c° is less than c; and it is worthy of remark, that we have only had recourse to a single new function, and not to two, as in the case of elliptic arcs, so that the result in the present case is much more simple. It remains to determine φ° by means of φ, but the assumed relation between these two angles, gives

$$\sin.\varphi^{\circ} = \frac{(1+b)\sin.\varphi\cos.\varphi}{\Delta},$$

$$\cos.\varphi^{\circ} = \frac{1-(1+b)\sin.^{2}\varphi}{\Delta},$$

$$\tan.\varphi^{\circ} = \frac{(1+b)\tan.\varphi}{1-b\tan^{2}.\varphi}.$$

This last result may be put under the very simple form,

$$\tan.(\varphi^{\circ}-\varphi) = b\tan.\varphi.$$

Let us therefore suppose, that after having determined the decreasing *moduli* c°, $c^{\circ\circ}$, &c. as also their complements b°, $b^{\circ\circ}$, &c. as has been explained in the preceding article, we calculate successively the amplitudes φ°, $\varphi^{\circ\circ}$, $\varphi^{\circ\circ\circ}$, &c. by the formulas

$$\begin{aligned}\tan.(\varphi^{\circ}-\varphi) &= b\ \tan.\varphi,\\ \tan.(\varphi^{\circ\circ}-\varphi^{\circ}) &= b^{\circ}\ \tan.\varphi^{\circ},\\ \tan.(\varphi^{\circ\circ\circ}-\varphi^{\circ\circ}) &= b^{\circ\circ}\ \tan.\varphi^{\circ\circ},\\ &\&c.\end{aligned}$$

which may also be done by the series

$$\varphi^{\circ} = 2\varphi - c^{\circ}\sin.2\varphi + \tfrac{1}{2}c^{\circ 2}\sin.4\varphi - \tfrac{1}{3}c^{\circ 3}\sin.6\varphi + \&c.$$

$$\varphi^{\circ\circ} = 2\varphi^{\circ} - c^{\circ\circ}\sin.2\varphi^{\circ} + \tfrac{1}{2}c^{\circ\circ 2}\sin.4\varphi - \tfrac{1}{3}c^{\circ\circ 3}\sin.6\varphi^{\circ} + \&c.$$

Then the integral $\int \frac{d\varphi}{\Delta}$, or the function F will be expressed thus,

$$F = \frac{1+c^{\circ}}{2} \cdot F^{\circ},$$

$$F = \frac{1+c^{\circ}}{2} \cdot \frac{1+c^{\circ\circ}}{2} \cdot F^{\circ\circ},$$

$$F = \frac{1+c^{\circ}}{2} \cdot \frac{1+c^{\circ\circ}}{2} \cdot \frac{1+c^{\circ\circ\circ}}{2} \cdot F^{\circ\circ\circ},$$

&c.

But when c is become very small, we have $\Delta = 1$, and $\int \frac{d\varphi}{\Delta} = \varphi$. Therefore, let Φ be the limit of the angles φ, $\frac{\varphi^{\circ}}{2}$, $\frac{\varphi^{\circ\circ}}{4}$, $\frac{\varphi^{\circ\circ\circ}}{8}$, &c. to which limit they approach extremely near after a few terms, and we have

$$F = \Phi\,(1+c^{\circ})\,(1+c^{\circ\circ})\,(1+c^{\circ\circ\circ}), \&c.$$

When $\varphi = 90^{\circ} = \frac{1}{2}\pi$, the limit Φ is also $\frac{1}{2}\pi$, so that the value of F, which is then F1, becomes

$$F1 = \frac{\pi}{2}\,(1+c^{\circ})\,(1+c^{\circ\circ})\,(1+c^{\circ\circ\circ}), \&c.$$

The constant product $(1+c^{\circ})\,(1+c^{\circ\circ})$ &c. which we shall call α, may be also represented by the expression

$$\alpha = \frac{2\sqrt{c^{\circ}}}{c} \cdot \frac{2\sqrt{c^{\circ\circ}}}{c^{\circ}} \cdot \frac{2\sqrt{c^{\circ\circ\circ}}}{c^{\circ\circ}}, \&c.$$

and under this form, it may easily be calculated by logarithms. The value of α being known, we have, in general $F = \alpha\Phi$ and $F1 = \alpha \cdot \frac{1}{2}\pi$.

28. The quickness of these approximations is best perceived in their applications. To render them as easy as possible, let it be observed, that if we make $c = \sin.\,\mu$, we have $b = \cos.\,\mu$, and $c^{\circ} = \tan.^{2}\,\frac{1}{2}\mu$, from which it appears that c° is easily deduced from c by the table of sines. Making in like manner $c^{\circ} = \tan.^{2}\,\frac{1}{2}\mu = \sin.\,\mu^{\circ}$, which gives a new angle μ°, we have $c^{\circ\circ} = \tan.^{2}\,\frac{1}{2}\mu^{\circ}$, and so on. When we at last come to a very small c, in order to find the next one, which I denote by c°, we may, to avoid small angles, employ the formula

$$c^{\circ} = \frac{1}{4}c^{2} + \frac{1\cdot3}{4\cdot6}\,c^{4} + \frac{1\cdot3\cdot5}{4\cdot6\cdot8} + c^{6}\ \&c.$$

of which the first term, or at most the first two, will be sufficient.

As to the calculation of the angles φ°, $\varphi^{\circ\circ}$, &c. we have nothing to add to the simplicity of the formula tan. $(\varphi^{\circ}-\varphi)=b$ tan. φ, which is well adapted to trigonometrical calculation; we shall only observe, that the angle $\varphi^{\circ}-\varphi$, thus determined by its tangent, is almost equal to the angle φ, when c becomes very small, for we have, without any ambiguity,

$$\varphi^{\circ}=2\varphi-c^{\circ}\,\text{sin.}\,2\varphi+\tfrac{1}{2}c^{\circ 2}\,\text{sin.}\,4\varphi-\tfrac{1}{3}c^{\circ 3}\,\text{sin.}\,6\varphi+\&c.$$

It is therefore necessary to take for the value of $\varphi^{\circ}-\varphi$, not always the smallest angle which is found from the table of sines, but that which approaches most to φ, and which may be several circumferences. An example will shew, that on this point there is neither difficulty nor ambiguity.

EXAMPLE.

Required the value of the function F, when $c=\tfrac{1}{2}\sqrt{(2+\sqrt{3})}$ and tan. $\varphi=\sqrt{\frac{2}{\sqrt{3}}}$. We purposely choose a case unfavourable to calculation, because the value of c differs but little from unity.

The following table shews how the values of c°, $c^{\circ\circ}$, b°, $b^{\circ\circ}$, &c. are calculated, according to what has been explained.

Values of c and b.						*Their logarithms.*
c	=	sin.	75°	0′	0″	9·9849438.
b	=	cos.	75	0	0	9·4129962.
c°	=	tan.² sin.	37 36	30 4	0 7·47	9·7699610.
b°	=	cos.	36	4	7·47	9·9075648.
$c^{\circ\circ}$	=	tan.² sin.	18 6	2 5	3·735 9·36	9·0250880.
$b^{\circ\circ}$	=	cos.	6	5	9·36	9·9975454.
$c^{\circ\circ\circ}$	=	tan.² sin.	3 0	2 9	34·68 42·90	7·4511672.
$b^{\circ\circ\circ}$	=	cos.	0	9	42·90	9·9999983.
$c^{\circ\circ\circ\circ}$	=	$\tfrac{1}{4}(c^{\circ\circ\circ})^2$				4·3002761.
$b^{\circ\circ\circ\circ}$						0·0000000.

These logarithms give readily the product $\alpha=\frac{2\sqrt{c^{\circ}}}{c}\cdot\frac{2\sqrt{c^{\circ\circ}}}{c^{\circ}}$, &c. for which four factors are sufficient, and we have log. $\alpha=0·2460561$, or $\alpha=1·7622037$. Hence we find directly $F1=\frac{\pi}{2}\cdot\alpha=2·7680632$. We next find

$$\begin{array}{llrrr}
\varphi & = & 47^{\circ} & 3' & 30''{\cdot}96. \\
\varphi^{\circ} & = & 62 & 36 & 3{\cdot}12. \\
\varphi^{\circ\circ} & = & 119 & 55 & 47{\cdot}69. \\
\varphi^{\circ\circ\circ} & = & 240 & 0 & 0{\cdot}18. \\
\varphi^{\circ\circ\circ\circ} & = & 480 & 0 & 0{\cdot}00. \\
\&c. & & & &
\end{array}$$

The other values of φ increase in a double ratio, from which it follows that the limit of the angles φ, $\frac{\varphi^{\circ}}{2}$, $\frac{\varphi^{\circ\circ}}{4}$, &c. is $=$ 30° or $\frac{\pi}{6}$, and thus we have $F = \frac{\pi}{6}\alpha = 0{\cdot}9226877$, as it art. 20.

Let us remark, that since the limit $\varphi = 30^{\circ} = \frac{1}{3}90^{\circ}$, we have $F = \frac{1}{3}F1$; this equality is at least approximate, since such is the value found by the calculation; but it is easy to be assured that it is accurately so; for it has appeared (art. 18.), that the equation to be resolved to make $F(\varphi) = \frac{1}{3}F1$ is

$$0 = 1 - 2 \sin.\varphi + 2c^2 \sin.^3\varphi - c^2 \sin.^4\varphi;$$

but in the present case where $c^2 = \frac{2+\sqrt{3}}{4}$ we shall find $\sin.\varphi = \sqrt{3} - 1$, or $\tan.\varphi = \sqrt{\left(\frac{2}{\sqrt{3}}\right)}$; therefore the function F is accurately one-third of $F(90^{\circ})$ or $F1$.

29. The value of φ being given, it appears that the function F may be easily found to the necessary degree of accuracy. On the other hand it may be useful to determine the amplitude φ, when the value of the function F is supposed known. Let α always denote the product $(1 + c^{\circ})(1 + c^{\circ\circ})$ &c. and Φ the limit of the angles φ, $\frac{\varphi^{\circ}}{2}$, $\frac{\varphi^{\circ\circ}}{4}$, &c. Seeing we have $F = \alpha\Phi$, it follows that $\Phi = \frac{F}{\alpha}$; thus the limit Φ is known. Let us suppose that the values of c°, $c^{\circ\circ}$, &c. have been calculated to a term so small as to admit of being neglected, and for example, let this term be $c^{\circ\circ\circ\circ\circ}$, or, for the sake of brevity, $c^{\circ 5}$. Since, therefore, $\varphi^{\circ 5} = 2\varphi^{\circ 4} - c^{\circ 5} \sin. 2\varphi^{\circ 4} +$ &c. we have $\varphi^{\circ 5} = 2\varphi^{\circ 4}$, very near, and much more $\varphi^{\circ 6} = 2\varphi^{\circ 5}$ &c. Therefore the limit Φ is equal to $\frac{1}{16}\varphi^{\circ 4}$, that is we have $\varphi^{\circ\circ\circ\circ} = 16\,\Phi$. This last φ being known, we return to those which precede it by the successive equations

$$\begin{aligned}
\sin.\ (2\varphi^{\circ\circ\circ} &- \varphi^{\circ\circ\circ\circ}) = c^{\circ\circ\circ\circ} \sin.\ \varphi^{\circ\circ\circ\circ}.\\
\sin.\ (2\varphi^{\circ\circ} &- \varphi^{\circ\circ\circ}) = c^{\circ\circ\circ} \sin.\ \varphi^{\circ\circ\circ}.\\
\sin.\ (2\varphi^{\circ} &- \varphi^{\circ\circ}) = c^{\circ\circ} \sin.\ \varphi^{\circ\circ}.\\
\sin.\ (2\varphi &- \varphi^{\circ}) = c^{\circ} \sin.\ \varphi^{\circ}.
\end{aligned}$$

These have all the same form, and the last of them is deducible from the equation tan. $(\varphi^{\circ} - \varphi) = b$ tan: φ.

This is particlarly applicable to the resolution of the equation $F(\varphi n) = nF(\varphi)$ whatever be the value of n. The angle φ being given we can find $F(\varphi)$ and $nF(\varphi)$; therefore $F(\varphi n)$ will be known, and thence φn may be found as has been explained. This method in no case requires many operations, whereas if n were somewhat great, or only fractional, the methods we have formerly given to determine φn may become very tedious, or even impracticable. Thus the present method is almost always preferable.

(To be continued in the next Volume.)

End of the Third Part of the Second Volume.

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